Directive Surface Plasmons on Tunable Two-Dimensional Hyperbolic Metasurfaces and Black Phosphorus: Green’s Function and Complex Plane Analysis

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Abstract—We study the electromagnetic response of two- and quasi-two-dimensional (2-D) hyperbolic materials, on which a simple dipole source can excite a well-confined and tunable surface plasmon polariton (SPP). The analysis is based on the Green’s function for an anisotropic 2-D surface, which nominally requires the evaluation of a 2-D Sommerfeld integral. We show that for the SPP contribution, this integral can be evaluated efficiently in a mixed continuous-discrete form as a continuous spectrum contribution (branch cut integral) of a residue term, in distinction to the isotropic case, where the SPP is simply given as a discrete residue term. The regime of strong SPP excitation is discussed, and the complex-plane singularities are identified, leading to physical insight into the excited SPP. We also present a stationary phase solution valid for large radial distances. Examples are presented using graphene strips to form a hyperbolic metasurface and thin-film black phosphorus. Green’s function and complex-plane analysis developed allows for the exploration of hyperbolic plasmons in general 2-D materials.

Index Terms—Anisotropy, complex plane analysis, directed surface plasmon, Green’s function, hyperbolic surface.

I. INTRODUCTION

RECENTLY, the development of nanofabrication technologies has made it possible to fabricate artificial materials exhibiting a hyperbolic regime—hyperbolic metamatals (HMTMs) [1], [2]. HMTMs are uniaxial structures with extreme anisotropy, whose reactive effective material tensor components have the opposite signs for orthogonal electric field polarizations [3]. Hyperbolic materials exhibit hyperbolic, as opposed to the usual elliptic, dispersion, and combine the properties of transparent dielectrics and reflective metals [1]. These exotic properties have led to new physical phenomena and to the proposal for optical devices for a wide range of applications, such as far-field subwavelength imaging, nanolithography, emission engineering [1], negative index waveguides [4], subdiffraction photonic funnels [5], and nanoscale resonators [6].

The complexity of bulk fabrication of metamaterials has hindered the impact of this technology, especially in the optical regime, and volumetric effects may be detrimental to the associated losses [3]. Metasurfaces [7], [8], sheets of material with extreme subwavelength thickness, might address many of the present challenges and allow integration with planarized systems compatible with integrated circuits. Many high frequency electronics applications are envisioned for metasurfaces due to their ability to support and guide highly confined surface plasmons. The class of 2-D atomic crystals [9] represents the ultimate embodiment of a metasurface in terms of thinness, and often performance (e.g., tunability, flexibility, and quality factor). Some notable examples of 2-D layered crystals include graphene, transition metal dichalcogenides, trichalcogenides, black phosphorus (BP), boron nitride, and many more.

Graphene in particular has received considerable attention as a promising 2-D surface for many applications relating to large enhancement in Purcell emission, integrability, electronic tenability, and tranformation optics [10]–[17]. In addition to graphene, BP is also a layered material, with each layer forming a puckered surface due to sp3 hybridization. It is one of the thermodynamically more stable phases of phosphorus, at ambient temperature and pressure [18]. BP has recently been exfoliated into its multilayers [19]–[22], showing good electrical transport properties. In particular, the optical absorption spectra of BP vary sensitively with thickness, doping, and light polarization, especially across the technologically relevant midinfrared to near-infrared spectrum [23]–[25]. Hence, it has also received considerable attention for optoelectronics, such as hyperspectral imaging and detection [26]–[29], photodetectors in silicon photonics [30], photoluminescence due to excitonic effects [31], and among many others.

Both natural materials and metasurfaces can be isotropic or anisotropic, and, e.g., isotropic graphene can be employed to form an effective anisotropic metasurface by modulating...
its conductivity [3], [14]. In addition, both natural materials and metasurfaces may exhibit a hyperbolic regime. The basic properties of plasmons on 2-D hyperbolic surfaces have been recently studied: for metasurfaces comprised of anisotropic plasmonic particles in [32], for graphene strips in [3], and for general continuum 2-D materials including BP in [33].

In this paper, we provide Green’s function for an anisotropic 2-D surface in the Sommerfeld integral form. We focus on complex-plane analysis of Green’s function for the surface plasmon polariton (SPP) contribution in the hyperbolic case. The nominally 2-D Sommerfeld integral form of Green’s function is very time-consuming to evaluate, and provides no physical insight into the resulting field. Here, we show that for the SPP field, this integral can be evaluated efficiently in a mixed continuous-discrete form as a continuous spectrum contribution (branch-cut integral) of a residue term. Complex-plane singularities are identified with various branch-cut integrals, leading to physical insight into the excited SPP. For some 2-D materials, the surface conductivity is rather weak, and a discussion is provided concerning the strength of the reactive conductivity response to maintain an SPP.

This paper is organized as follows. We discuss Green’s function calculation for an anisotropic 2-D sheet with conductivity tensor $\sigma$. A Hertzian dipole vertical current source serves as the geometry under consideration is shown in Fig. 1. The Sommerfeld dyadic is found to have the form

$$\pi^{(1)}(r) = \pi^{(2)}(r) = \int_{\Omega} [g^p(r', r') + g^t(r, r')] \cdot \frac{J^{(1)}(r')}{i\omega \epsilon_1} d\Omega'$$

where the underscore indicates a dyadic quantities, $g^p$ is the principal (free space) dyadic Green’s function, $g^t$ is the reflected dyadic Green’s function responsible for the fields in the region containing the source, $\pi^{(1)}$ is the transmitted dyadic Green’s function responsible for the fields in the nonsource region (here, we assume a source in one region or the other, but not in both regions), and $\Omega$ is the support of the current. With $y$ parallel to the interface normal, the principle Green’s dyadic can be written as

$$g^p(r, r') = \frac{1}{4\pi R} e^{-ik_r R}$$

$$\quad = \frac{1}{2} \int_{-\infty}^{\infty} e^{-p|y-y'|} \frac{1}{p_1} e^{-i|q-q'|} dq_x dq_z$$

where $q = q_x + i q_z$, $|q| = q = (q_x^2 + q_z^2)^{1/2}$, $p_2 = q_x^2 - k^2$, $\rho = ((x-x')^2 + (z-z')^2)^{1/2}$, $R = |r-r'| = (\rho^2 + (y-y')^2)^{1/2}$, and $1$ is the unit dyadic. The scattered (reflected or transmitted) Green’s dyadics can be obtained by enforcing the boundary conditions

$$\bar{z} \times (H_1 - H_2) = J^t_e$$

$$\bar{z} \times (E_1 - E_2) = -J^m_e$$

where $J^t_e$ (A/m) and $J^m_e$ (V/m) are electric and magnetic surface currents on the boundary. In our case, $J^m = 0$, and $J^t = \sigma \cdot E$. Using only an electric Hertzian potential, we can satisfy Maxwell’s equations and the relevant boundary conditions. Introducing the 2-D Fourier transform

$$a(q, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(r) e^{iqr} dxdz$$

$$a(r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a(q, y) e^{-iqr} dq_x dq_z$$

and enforcing the boundary conditions, the scattered Green’s dyadic is found to have the form

$$g^{r,t}_\alpha = \begin{pmatrix} g_{x,t}^{r,t} & g_{y,t}^{r,t} & 0 \\ g_{x,y}^{r,t} & g_{y,y}^{r,t} & g_{y,t}^{r,t} \\ 0 & g_{y,t}^{r,t} & g_{z,z}^{r,t} \end{pmatrix}$$

where the Sommerfeld integrals are

$$g^{r,t}_\alpha(q_x, q_z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^{r,t}_{\alpha}(q_x, q_z) e^{-p_1|y|} dq_x dq_z.$$
Green’s dyadic for region 2, $\mathbf{g}(\mathbf{r}, \mathbf{r}')$, has the same form as for region 1, although in (10), the replacement $w_{\alpha\beta}^e e^{-p_i(y+y')}$ → $w_{\alpha\beta}^l e^{p_2 y} e^{-p_2 y'}$ must be made.

The coefficients $w_{\alpha\beta}^e$ are complicated for the inhomogeneous case, and so for simplicity in the following, we assume that the sheet is in a homogeneous space $\varepsilon_2 = \varepsilon_1 = \varepsilon$, $\mu_2 = \mu_1 = \mu$. When region 2 differs from region 1, the only change is in the functions (11) and (12) provided in the following. Concentrating on the field in the upper half-space, $w_{\alpha\beta}^e = N_{\alpha\beta}(q_x, q_z)/D(q_x, q_z)$, where

$$D(q_x, q_z) = 2\sigma_{xx}(k^2 - q^2_z) + 2\sigma_{zz}(k^2 - q^2_x) - i\frac{4k}{\eta} \left(1 + \frac{1}{4}\eta^2 \sigma_{xx}\sigma_{zz}\right),$$

and

$$N_{yy}(q_x, q_z) = -p^2 (\sigma_{xx} + \sigma_{zz}) - ipk\eta\sigma_{xx}\sigma_{zz}$$

$$N_{xy}(q_x, q_z) = iq_p p(\sigma_{xx} - \sigma_{zz})$$

$$N_{zy}(q_x, q_z) = -iq_p p(\sigma_{xx} - \sigma_{zz})$$

(12)

where $p = (q_x^2 + q_z^2 - k^2)^{1/2}$, and $\eta = (\mu/\varepsilon)^{1/2}$. Then, e.g., for the vertical field in the upper half-space

$$E_y = \frac{1}{i\omega\varepsilon} \left(k^2 + \frac{\partial^2}{\partial y^2}\right) \left(g_{yy}^p(r, r') + g_{yy}^r(r, r')\right) + \frac{1}{i\omega\varepsilon} \left(\frac{\partial^2}{\partial x\partial y} g_{xy}^p(r, r') + \frac{\partial^2}{\partial z\partial y} g_{zy}^r(r, r')\right)$$

and other field components are obtained from (2).

III. DIRECTIONAL PROPERTIES OF SPPS ON 2-D SURFACES

Before considering complex-plane evaluation of Green’s functions, we describe some basic properties of SPPs on hyperbolic 2-D surfaces [3], [32], [33]. In order to understand the behavior of surface waves, it is instructive to inspect the plasmon dispersion relation $D(q_x, q_z) = 0$ arising from (11), the denominator of Green’s function. As we show later, in the general case, SPPs are obtained as a mixture of TE and TM modes, and moreover, it is not possible to solve for the wave vector eigenmodes $q_x$ and $q_z$ from the single complex-valued equation (11). Furthermore, unlike for isotropic surfaces, for an anisotropic medium, the direction of energy transfer is defined by the group velocity in the medium [34] $\nabla_\theta \omega(q)$, and does not coincide with the direction of the plasmon wave vector $\mathbf{q}$. In our case, the dispersion relation for surface plasmons is complicated and the group velocity cannot be calculated analytically. However, we can estimate the direction of plasmon propagation geometrically by examining the plasmon’s equifrequency contours, $\omega(q) = \text{const}$. As the group velocity is a gradient of frequency with respect to wave vector, the direction of plasmon energy flow is necessary orthogonal to the equifrequency contours.

Assuming that the conductivity is purely imaginary and lossless, $\sigma_{jj} = i\sigma''_{jj}$, $j = x, z$, and that $q_x, q_z \gg k$, the zeros of (11) can be approximated as the solution of

$$\frac{q_x^2}{\sigma''_{xx}} + \frac{q_z^2}{\sigma''_{zz}} = 2\rho_0 \left(\frac{\varepsilon_0}{\sigma''_{xx}\sigma''_{zz}} - \frac{\mu_0}{4}\right).$$

(14)

Although the right side varies with $\mathbf{q}$ because of the square root $p$, the variation is less than the left side, and we can approximate the right side as being constant in wavenumber. Then, in the hyperbolic case ($\sigma''_{xx}, \sigma''_{zz} < 0$), the equifrequency surface (EFS) is a hyperbola, as shown in Fig. 2 for two values of surface conductivity [blue lines: $\sigma_{xx} = 0.003 + 0.25i$ mS and $\sigma_{zz} = 0.03 - 0.76i$ mS [blue hyperbola, see also Fig. 10(b)], and $\sigma_{xx} = 1.3 + 16.9i$ mS and $\sigma_{zz} = 0.4 - 9.2i$ mS [green hyperbola, see also Fig. 10(c)]] for comparison, the isotropic case for $\sigma_{xx} = \sigma_{zz} = 0.03 - 0.76i$ mS (black circle) is also shown. Red dashed line: $45^\circ$ with respect to the $x$-axis for guidance.

In the nonhyperbolic (purely anisotropic) case ($\sigma''_{xx}, \sigma''_{zz} > 0$), (14) is the equation for an ellipse in $\mathbf{q}$-space with the axis oriented along the $q_x$-axis and the $q_z$-axis. The length of the ellipse’s principal axes along the $q_x$-axis and the $q_z$-axis is proportional to $\sigma''_{zz}$ and $\sigma''_{xx}$, respectively. Thus, the EFS has a quasi-elliptic form elongated along the direction of the smallest component of the conductivity tensor, the degree of elongation being set by the ratio of $\sigma''_{xx}$ and $\sigma''_{zz}$. Later, in Fig. 11, we consider BP having $\sigma_{xx} = 0.0008 - 0.2923i$ mS and $\sigma_{zz} = 0.0002 - 0.0658i$ mS. Due to the strong elongation of the EFS along the $q_z$-axis,
the group velocity points approximately along the \( q_y \)-axis, such that the SPP carries energy along the \( x \) crystallographic axis (see Fig. 11).

IV. COMPLEX-PLANE ANALYSIS IN THE \( x \) PLANE

In the case of an isotropic material, the coefficients \( w_{\alpha\beta} \) only depend on \( q^2 - q_y^2 + q_z^2 \), leading to

\[
g_{\alpha\beta}^w(\mathbf{r}, \mathbf{r'}) = \frac{1}{2\pi} \int_{\gamma_1} w_{\alpha\beta}^{(p)}(q) e^{-ip(q+y')} \frac{J_0(q\rho q)dq}{2p} = \frac{1}{2\pi} \int_{-\infty}^{\infty} w_{\alpha\beta}(q) e^{-ip(q+y')} H_0^{(2)}(q\rho q)dq \tag{15}\]

where \( J_0 \) and \( H_0^{(2)} \) are the usual zeroth-order Bessel and Hankel functions, respectively. These two forms can be converted one into another using the relation \( J_0(\alpha) = (1/2)[H_0^{(1)}(\alpha) + H_0^{(2)}(\alpha)] \). \( H_0^{(2)}(-\alpha) = -H_0^{(1)}(\alpha) \).

In this case, such as occurs for graphene without a magnetic bias, the pole of \( w_{\alpha\beta} \) leads to a simple analytical form for the SPP field \([12]\). However, this is not the case for an anisotropic surface. Since the 2-D Sommerfeld integral can be time-consuming to evaluate, writing

\[
g_{\alpha\beta}^w(\mathbf{r}, \mathbf{r'}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq_x e^{-iq_x(z-z')} f_{\alpha\beta}^{\text{SPP}}(q_x) \tag{16}\]

where

\[
f_{\alpha\beta}^{\text{SPP}}(q_x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} w_{\alpha\beta}(q_x, q_z) e^{-ip(q+y')} \frac{e^{-iq_x(x-x')}}{2p} dq_x \tag{17}\]

the “inner” integral \( f_{\alpha\beta}(q_x) \) can be evaluated as an SPP residue term (discrete spectral component) and branch-cut integral representing the radiation continuum into space (note that the choice of “inner” and “outer” integrals is arbitrary). The branch cut in the \( q_x \) plane is the usual hyperbolic branch cut associated with the branch point due to \( p = (q_x^2 + q_z^2 - k^2)/2 \), occurring at \( q_x = \pm (k^2 - q_z^2)^{1/2} \) \([39]\]. Then

\[
f_{\alpha\beta}(q_x) = -iw_{\alpha\beta}^{\text{SPP}}(q_x, q_z) e^{-ip(q+y')} \frac{e^{-iq_x(x-x')}}{2p} + \frac{1}{2\pi} \int_{b c} w_{\alpha\beta}(q_x, q_z) e^{-ip(q+y')} \frac{e^{-iq_x(x-x')}}{2p} dq_x \tag{18}\]

where the first term is the residue contribution and \( b c \) indicates the hyperbolic branch-cut contour. In \( (18) \), \( \omega_{\alpha\beta}^{\text{SPP}}(q_x, q_z) = N(q_x, q_z)/D'(q_x, q_z), D'(q_x, q_z) = (\partial/\partial q_x)D(q_x, q_z), \) and where \( q_x \) is the root of \( D(q_x, q_z) = 0 \) for a given \( q_z \)

\[
q_x(q_z) = \pm \sqrt{-B \pm \sqrt{B^2 - 4AC}} \tag{19}\]

where \( A = \sigma_{xx}^2, B = (1/4)\sigma_y^2 - 2k^2\sigma_z^2 + 2(q_z^2 - k^2)\sigma_{xx}\sigma_{zz}, C = k^2(\sigma_{xx} + \sigma_{zz})^2 + q_z^2(q_z^2 - 2k^2)\sigma_{zz}^2 - 2k^2q_z\sigma_{xx}\sigma_{zz} + (1/4)\sigma_y^2(k^2 - k^2), \) and \( \alpha = (4k/\pi)(1 + (1/4)\eta^2)\sigma_{xx}\sigma_{zz} \).

When the SPP field is the dominant contribution to the response, which is the usual regime for plasmonics where the field close to the interface, \( (y, y' \ll \lambda) \) is of interest, the branch-cut term can be ignored, and the residue term suffices for the calculation of \( f(q_x) \)

\[
f_{\alpha\beta}^{\text{SPP}}(q_x) \approx -iw_{\alpha\beta}^{\text{SPP}}(q_x, q_z) e^{-ip(q+y')} \frac{e^{-iq_x(x-x')}}{2p} \tag{20}\]

which considerably speeds up evaluation of Green’s function (rendering it 1-D). Since \( q_x \) is the propagation constant along the \( x \)-axis, the \( \mp \) outside the square root in \( (19) \) indicates forward/backward propagation, whereas the inner \( \pm \) sign choice governs propagation of different modes (only one of which will propagate). Assuming \( (x-x') > 0 \), the term \( e^{-iq_x(x-x')} \) necessitates that \( \text{Im}(q_x) < 0 \) to have a decaying wave traveling away from the source along the \( x \)-axis.

As an example, we consider an anisotropic surface with \( \sigma_{xx} = 0.02 + 0.57i \) mS and \( \sigma_{zz} = 0.02 - 0.57i \) mS. As discussed in Appendix A, such a conductivity tensor can be physically realized by an array of densely packed graphene strips at terahertz and near-infrared frequencies. Fig. 3 compares \( f_{yy}(q_z) \) obtained numerically by performing the integral \( (16) \) and obtained by using the residue term only \([\text{see } (20)]\). The source is located at \( y' = \lambda/50 \), very near the surface, and radiating at frequency 10 THz. Clearly, in the SPP regime, the residue provides the dominant component of the response, and the branch-cut integral can be ignored. Although not shown, for source or observation points relatively far from the surface, the branch-cut integral is important, and can be the dominant contribution to the scattered field.

In the following, we are interested in surfaces that provide a strong reactive and low-loss response, \( \text{Im}(\sigma_{xx}) \gg \text{Re}(\sigma_{xx}) \). In addition to this inequality, \( \text{Im}(\sigma_{xx}) \) must not be too small \([40]\). The ability of a surface to support a strong SPP depends on the ratio of the branch cut term (space radiation spectra) to the residue (SPP) term in the inner integral \( (18) \). In Fig. 4, we assume a general hyperbolic form \( \sigma_{xx} = \sigma_0(0.01 + i) \) and \( \sigma_{zz} = 0.1\sigma_{xx}^* \), where \( \sigma_0 = c^2/4\hbar \) is the conductance quantum, \( c \) is the electron charge, and \( * \) indicates complex conjugation. We assume that losses are relatively small, and use \( \alpha \) in order to vary the magnitude of the conductivity.

It is clearly shown in Fig. 4 that for conductivity values smaller than the conductance quantum, the radiation spectra is dominant (in the limit that \( |\sigma_0| \to 0 \), the surface vanishes...
Since in this case, $q$ is directed along a specific angle. For the isotropic case therefore, in the anisotropic hyperbolic case, the SPP wave is radially propagating along all directions in the plane of the surface. In this case, as shown in (22) and (24), energy propagation is focused along the specific directions governed by the conductivity components [3].

V. APPROXIMATION OF THE OUTER INTEGRAL USING STATIONARY PHASE AND EXACT EVALUATION USING THE CONTINUOUS SPECTRUM

Although the SPP field can be evaluated from a numerical 1-D integral, (16) with (20), it is useful to consider other methods of evaluation that are more computationally rapid, and which lead to physical insight into the problem.

A. Stationary Phase Evaluation of the Outer Integral

The “outer” integral (16) using (20) can be approximated by the well-known method of SP [41]. In particular, an analysis similar to that needed here was performed in [42], where the inner integral is approximated as a residue (ignoring the branch-cut contribution, as we do here), and the outer integral is evaluated using SP. Regarding computation of the outer integral, although it seems difficult to show analytically because of the complicated expression (19) for the pole $q_{SP}(q_{z})$, the numerical tests show that $\text{Re}(q_{z}^{2}p_{s}^{2} + q_{z}^{2} - k_{s}^{2}) > 0$ for small values of $q_{z}$. Therefore, no leaky waves are encountered for typical parameter values.

SP evaluation of (16) with (20), assuming $\rho \gg (y + y')$, results in, to the first order

$$g_{SP}^{e}_{x}(r, r') = \sqrt{\frac{i}{2\pi \gamma'(q_{s}) \mu_{0}}} w_{SP}^{e}(q_{s}) e^{-\rho(q_{s})(y+y')} e^{-i\gamma(q_{s})}$$

(25)

where $w_{SP}^{e}(q_{s}) = w_{SP}^{e}(q_{s}q_{s}(q_{s}), q_{s})$, $p(q_{s}) = p(q_{s}q_{s}(q_{s}), q_{s})$, and $\gamma(q_{s}) = -q_{s}(q_{z}(q_{s})(x - x') + q_{z}(z - z'))$, which is the root of $d\gamma/dq_{z} = 0$, which can be obtained as the root of a fourth-order polynomial, or via numerical root search. See [42] for a ray-optical interpretation of the SP result in anisotropic media.
Although the main numerical results will be presented in Section VI, here we provide a comparison between the SP result (25) and numerical (real-line) computation of the outer integral (16). Fig. 5 shows the SP result (red) and numerical integration result (blue) for \( \sigma_{xx} = 0.02 + 0.57i \) mS and \( \sigma_{zz} = 0.02 - 0.57i \) mS and \( \sigma_{zz} = 0.003 + 0.25i \) mS, \( \sigma_{zz} = 0.03 - 0.76i \) mS, \( \rho = 0.4 \), \( \rho/(y + y') = 80 \), and \( f = 10 \) THz.

B. Complex-Plane Analysis in the \( q_z \) Plane

Although the SPP field can be evaluated to first-order using the SP approximation for \( \rho/(y + y') \gg 1 \), it is useful to consider complex-plane analysis of the “outer” integral over \( q_z \), which turns out to involve only continuous spectrum. This method is theoretically exact, and is valid for all field and source points. Furthermore, it does not require finding the \( q_z \) root, but does require knowing the \( q_z \) plane branch points and cuts, which, themselves, lead to considerable physical insight.

The Weierstrass preparation theorem shows that the complex function \( f^{(p)}_{SPP}(q_z) \), (20), has no poles, only branch points. Regarding the two complex planes \( q_z - q_z \), a sufficient condition in order to have a branch point in the \( q_z \) is that [43], [44]

\[
D(q_x, q_z) = \frac{\partial}{\partial q_z} D(q_x, q_z) = 0
\]

with \( \delta = (\partial/\partial q_z) D(q_x, q_z) (\partial^2/\partial q_z^2) D(q_x, q_z) \neq 0 \). Although (26) represents a second-order zero of \( D \), in the \( q_z \)-plane, these points are not poles, and are also not necessarily \( q_z \) plane branch points without the condition \( \delta \neq 0 \). These branch points are associated with modes in the \( q_z \) plane merging at a certain value of \( q_z \), forming a second-order zero of \( D \). Thus, the pair \( (q_z, q_z) \) satisfying (26) and \( \delta \neq 0 \) represent poles in the \( q_z \) plane and branch points in the \( q_z \) plane (the branch in the \( q_z \) plane controls the merging of poles in the \( q_z \) plane). Another possible branch point in the \( q_z \) plane is associated with the square root in \( p \). The fact that a pole in one spectral plane results in a branch point in another spectral plane was recognized in studies of microstrip and other integrated waveguides [35]–[38]. It is also worthwhile to note that the asymptotic methods for branch-cut evaluation described in [41] do not work here. To use those formulas, the branch-cut integral must be dominated by the branch point, that is, by the section of the integral in the vicinity of the branch point. This is not the case for the anisotropic problem, where we have found that the sections of the branch-cut integral far from the branch point can contribute substantially.

C. p-Type Branch Point in the \( q_z \) Plane

For the isotropic case, \( p = (q_z^2 - k^2)^{1/2} \) and the p-type branch point occurs at \( q = \pm k \), resulting in the usual hyperbolic branch cuts in the \( q_z \) plane [39]. In this case, \( q_z^2 + q_x^2 = q_{p2}^2 \) is a constant and \( q_z = (k^2 - q_{p2}^2)^{1/2} \) leads to branch points at \( q_z = \pm k \). However, for the residue, \( q_{p2}^2 = q_{p2}^2(q_z) + q_z^2 \) is a constant in \( q_z \) and so we never have \( q_{p2} = k \) for any \( q_z \), and so there is no p-type BP in the \( q_z \) plane for the SPP for the isotropic case. However, for anisotropic media, \( q_{p2}^2(q_z) + q_z^2 \) is not generally a constant, and so there can be a “p-type” BP in the \( q_z \)-plane, where \( p = (q_{p2}(q_z) + q_z^2 - k^2)^{1/2} = 0 \), although this will not occur at \( q_z = k \) unless \( q_{p2}(k) = 0 \). In any event, since this branch cut relates to radiation into space, for the SPP, we can ignore this contribution to the SPP field.

Introducing the notation that \( (q_z^{(n)}, q_z^{(n)}) \) represents the pair of spectral values that satisfy the conditions for a branch point/pole pair, (26) and \( \delta \neq 0 \), since the residue term already satisfies \( D(q_{p2}, q_z) = 0 \), and we can find branch points in the \( q_z \) plane from

\[
\left( \sigma_{xx} + \frac{ik/\eta}{\sqrt{q_{p2}^2 + q_z^2 - k^2}} \left( 1 + \frac{4\eta^2\sigma_{xx}\sigma_{zz}}{\eta^2\sigma_{xx}} \right) \right) q_{p2}(q_z) = 0.
\]

As we will show later, these branch points have a significant role in the analysis of the SPP. Because of their importance, we categorize them into two groups, type-0 and type-1 branch points.

D. Type-0 Branch Point in the \( q_z \) Plane

First, we define type-0 branch points as those values of \( q_z \) for which \( q_{p2}(q_z) = 0 \) in (27), i.e., the merging of the forward and backward modes [associated with different signs in the outer square root in (19)] in the \( q_z \) plane at a certain value of \( q_z \) [44], given by

\[
q_{z(0)} = q_z^{(0)} = k \sqrt{1 - \frac{2}{\eta \sigma_{zz}}} \quad (28)
\]

\[
q_{z(-0)} = q_z^{(-0)} = k \sqrt{1 - \frac{\eta \sigma_{xx}}{2}} \quad (29)
\]

such that the pair \( (q_x, q_z) = (0, q_z^{TM/TE}) \) form a pole-branch-point pair. For \( \sigma_{xx} = \sigma_{zz} \), these are well-known TM and TE SPP wavenumbers, respectively (graphene is an example of such a 2-D isotropic layer, which can support
these modes [12]). Note that for isotropic media, a vertically polarized current source will produce only TM fields (although a horizontally polarized source will produce both TE and TM fields even when the sheet is isotropic [39]). For an anisotropic sheet, the boundary conditions cannot be satisfied assuming only one type of field.

E. Type-1 Branch Point in the $q_z$ Plane

Another set of singularities in the $q_x$–$q_z$ plane is related to the point in the $q_z$ plane where modes $q_{TM}$ associated with different signs in the inner square root in (19) merge for $q_{xp} \neq 0$. These can be obtained by simultaneously solving the equations $D(q_x, q_z) = 0$ and $(dD(q_x, q_z)/dq_x) = 0$, leading to

$$q_z^{(1)} = \frac{-k^2}{\delta \sigma} \left( \sigma_{xx} + (\sigma_{zz} - 2\sigma_{xx}) \left( 1 + \frac{1}{2} \eta^2 \sigma_{xx} \sigma_{zz} \right) \right)^{-1} \sigma_{zz}$$

(30)

$$q_z^{(2)} = \sqrt{-\left(q_x^{(2)}\right)^2 + k^2 \left( 1 - \frac{1}{2} \eta^2 \sigma_{xx} \sigma_{zz} \right)^2}$$

(31)

where $\delta \sigma = \sigma_{zz} - \sigma_{xx}$, such that $(q_x, q_z) = (q_x^{(1)}, q_z^{(1)})$ forms a pole-branch-point pair.

F. Branch-Cut Analysis in the $q_z$ Plane

Using the SPP field (20) and performing the outer integration, Green’s function is

$$g_{ab}^r = \frac{-i}{2\pi} \int_{-\infty}^{+\infty} w_{ab}(q_{xp}, q_z) \frac{e^{-\rho(y-y')}}{2p} e^{-i q_{xp}(x-x')} \times e^{-i q_z(z-z')} \, dq_z.$$  

(32)

Assuming $(z-z') > 0$, due to the term $e^{-i q_z(z-z')}$, the contour can be closed in the lower half-plane of the $q_z$ plane, leading to

$$g_{ab}^r \approx \frac{-i}{2\pi} \int_{bc} w_{ab}(q_{xp}, q_z) \frac{e^{-\rho(y+y')}}{2p} e^{-i q_{xp}(x-x')} \times e^{-i q_z(z-z')} \, dq_z.$$  

(33)

where the branch-cut integral is over all branch cuts. Also, from the term $e^{-i q_{xp}(x-x')}$, it is clear that for $x - x' \geq 0$ then only when $\text{Im}(q_{xp}) \leq 0$ do we obtain an SPP that decays away from the source. Therefore, we have in the $q_z$ plane two Riemann sheets (as mentioned previously, neglecting the $p$-type branch point, which would introduce another two sheets; here, we simply enforce $\text{Re}(\rho) > 0$), the top (proper) sheet where $\text{Im}(q_{xp}) \leq 0$ and the bottom sheet where $\text{Im}(q_{xp}) \geq 0$, for $x - x' \geq 0$. Those values of $q_z$ that lead to $\text{Im}(q_{xp}) = 0$ determine the branch-cut trajectory, which separates the proper from improper Riemann sheets.

Typically, branch-cut trajectories to separate certain Riemann sheets can be analytically determined from the functional dependence of the multivalued function that defines the branch point. However, for anisotropic surfaces, the form of $q_{xp}$ is too complicated to determine a simple equation for the branch cut for $\text{Im}(q_{xp}) = 0$. As an example,
such as graphene in the far-infrared the lossless BC contour is sufficient. (numerical example considered).

As a common special case, for an inductive isotropic surface such as graphene in the far-infrared

$$q_{z}^{TM} \text{ and } q_{z}^{-1} \text{ (between } q_{z} = 1.005k \text{ and } -3.22i k \text{ in the numerical example considered).}

The lossy case is shown in Fig. 7; the branch cut trajectory deflects a bit from the lossless case, but for low-loss surface, the lossless BC contour is sufficient.

As a common special case, for an inductive isotropic surface such as graphene in the far-infrared

$$\sigma_{xx} = \sigma_{zz} = \frac{-i e^{2} k_{B} T}{\pi \hbar^{2} (\omega - i 2 \Gamma)} \times \left(\frac{\mu_{c}}{k_{B} T} + 2 \ln \left(1 + e^{-\frac{\mu_{c}}{k_{B} T}}\right)\right). \quad (36)$$

Here, we consider graphene at $T = 300$ K, $\mu_{c} = 0.5$ eV, and $f = 20$ THz. In this case, the TE related branch point is at $q_{z}^{TE} = k(1.0039 + 0.0001i)$, and so is not implicated in the lower half-plane closure, consistent with the surface being inductive (no TE mode is supported). Since only TM branch points occur, only a TM mode exists, and the TM-related BP occurs at $q_{z}^{TM} / k = (11.3706 - 0.2088i)$. The two other type-1 branch points move to infinity as the surface becomes isotropic, and therefore, the branch cut extends down the entire imaginary axis (therefore, for both the isotropic and anisotropic cases, there is a branch cut between $q_{z}^{TM}$ and $q_{z}^{-1}$). Fig. 8 shows a surface plot of $\text{Im}(q_{z})$ in the $q_{z}$ plane.

For isotropic and inductive graphene, only a TM mode can propagate, and so the contribution is from the TM-related branch point and associated cut, as expected. For the graphene strip array anisotropic case, the hybrid nature of the modes supported by such a surface involve both TE and TM-related branch points, and in contrast to the isotropic case, three branch points contribute to the field.

G. Conductivity and Its Effect on Branch Points and SPP Confinement

Analytically, it can be shown that both type-1 branch points $q_{z}^{(\pm 1)}$ can be connected to a TE or TM branch point, depending on the conductivity value. Two cases are of particular interest, small conductivity values, $(\text{Im}(\sigma_{xx/zz}))^{2} \ll 1$, and large conductivity values, $(\text{Im}(\sigma_{xx/zz}))^{2} \gg 1$. For small conductivity values, from (28) and (29), we have

$$q_{z}^{TM} = k \sqrt{1 - \left(\frac{2}{\eta \sigma_{zz}}\right)^{2}} \rightarrow (\eta \sigma_{zz})^{2} = \frac{4}{1 - \left(\frac{q_{z}^{TM}}{k}\right)^{2}} \quad (37)$$

$$q_{z}^{TE} = k \sqrt{1 - \left(\frac{\eta \sigma_{xx}}{2}\right)^{2}} \rightarrow \frac{1}{(\eta \sigma_{xx})^{2}} = \frac{1}{4} \frac{1}{1 - \left(\frac{q_{z}^{TE}}{k}\right)^{2}}. \quad (38)$$

Making these replacements in (30) and (31) and using the fact that for small conductivity like in our previous numeric example ($\sigma_{xx} = 0.02 + 0.57i$ mS and $\sigma_{zz} = 0.02 - 0.57i$ mS), we have $(\text{Im}(\sigma_{xx/zz}))^{2} \ll 1$, and then, $|q_{z}^{TM}| \gg k$ and $|q_{z}^{TE}| \approx k$, and so $|q_{z}^{TM}|^{2} \ll |q_{z}^{TM}|^{2}$, such that

$$q_{z}^{(\pm 1)} = k \sqrt{\frac{1}{1 - \left(\frac{q_{z}^{TM}}{k}\right)^{2}} \frac{\sigma_{xx} \mp 2 \sigma_{xx}}{\sigma_{zz} - \sigma_{xx}}}. \quad (39)$$

Therefore, for small values of $\sigma_{xx}$ and $\sigma_{zz}$, the type-1 branch points are governed by (and associated with) the TE branch point $q_{z}^{TE}$.

For larger values of $\sigma_{xx}$ and $\sigma_{zz}$, the situation is different. In this case, for $(\text{Im}(\sigma_{xx/zz}))^{2} \gg 1$ we have $|q_{z}^{TM}|^{2} \ll |q_{z}^{TM}|^{2}$ and it can be shown that an approximate expression for the type-1 branch point is (39) with $q_{z}^{TM}$ replacing $q_{z}^{TE}$; the type-1 branch points are associated with the TM-related branch point. As the conductivity changes from a small to a large value, $q_{z}^{TM}$ and $q_{z}^{TM}$ move toward each other and then cross, and eventually interchange roles. Setting (28) and (29) equal to each other, it can be shown that these type-0 branch points meet at a frequency, such that

$$\sigma_{xx} \sigma_{zz} = 4 / \eta^{2}.$$

As an example of a large conductivity situation, conductivity tensor components $\sigma_{xx} = 1.3 + 16.9i$ mS and $\sigma_{zz} = 0.4 - 9.2i$ mS are attainable using multilayer graphene to form the strip array. For this set of conductivities, the branch points and the branch cuts are shown in Fig. 9. As can be seen, $q_{z}^{TE}$ exceeds $q_{z}^{TM}$, there is a branch cut from $q_{z}^{TE}$ to infinity, a branch cut between $q_{z}^{TM}$ and $q_{z}^{-1}$, and $q_{z}^{-1}$ is connected to $q_{z}^{TM}$.
A. Anisotropic Hyperbolic Layer (Graphene Strip Array)

As shown in Appendix A, conductivity components $\sigma_{xx} = 0.02 + 0.57i$ mS and $\sigma_{zz} = 0.02 - 0.57i$ mS can be realized using an array of graphene strips with $\mu_c = 0.33$ eV, strip width $W = 59$ nm, and period $L = 64$ nm. For this anisotropic hyperbolic surface, Fig. 10(a) shows the electric field $E_y$, the dominant field component, computed as a real-line integral (32), and as a sum of branch cut integrals (33); excellent agreement is found between the two methods (the branch-cut integrals are faster to compute than the brute-force numerical integrals, but no attempt was made to optimize either integration). The branch cuts for this case are shown in Fig. 7. Fig. 10(b) and (c) shows similar agreement for different strip configurations as discussed in the following.

Although the direction of the beam is electronically controllable via the chemical potential, different combinations of physical parameters of the graphene strip array (width $W$ and periodicity $L$) can also be used to produce a desired beam. An optimum geometry to produce a beam in a certain direction can be found by tuning all of these parameters simultaneously.

From (22), in the hyperbolic regime, propagation along a desired direction can be obtained if the tensor conductivity components have the proper ratio. Designing a hyperbolic metasurface to produce a beam in a desired direction (e.g., choosing the strip width and period) can be done by trial-and-error tuning of all geometrical and electrical parameters of the system, but a multi-variable optimization, such as a genetic algorithm (GA) is a good choice for this task [48], [49]. Ideally, the physical layout of the metasurface (graphene strips in the case) should be designed so that the effective (homogenized) conductivity tensor elements are hyperbolic, and have large imaginary part and small real part, since such a surface can support a well-confined, long-range SPP. Here, we used the cost function to be minimized as

$$\Psi(L, W, \mu_c, \phi) = \alpha(\text{Re}(\sigma_{xx}) + \text{Re}(\sigma_{zz}))$$

$$+ \frac{\beta}{|\text{Im}(\sigma_{xx})| + |\text{Im}(\sigma_{zz})|} + \gamma \left(\tan^2(\phi) + \frac{\sigma_{zz}}{\sigma_{xx}}\right)$$

where $\sigma_{xx}$ and $\sigma_{zz}$ are defined in (41) in Appendix A. The cost function in (40) is a multiobjective cost function and the coefficients $\alpha$, $\beta$, and $\gamma$ assign a weight (0 to 1) to each objective regarding to its importance. The first term in (40) assures a large real part of conductivity, the second term assures a small real part of conductivity, the second term assures the correct ratio for $\sigma_{zz}$ and $\sigma_{xx}$ to obtain the SPP beam in desired direction specified by $\phi$. It was found that $\alpha = 0.2$ and $\beta = \gamma = 0.4$ lead to good results.

The physical strip geometry leading to the beam in Fig. 10(a) was found in this manner, for a specified beam angle of 45°. Note the excellent agreement between desired and obtained beam angle. The chemical potential was then changed to produce the beam at 52°, for a fixed geometry. Thus, a significant aspect of using a graphene strip array is its electronic tunability by, e.g., varying the bias to control the chemical potential.

In Fig. 10(b), a desired beam angle of 60° was sought, and the GA was used to determine the optimized parameters; $\mu_c = 0.45$ eV, $W = 56.1$ nm, and $L = 62.4$ nm, such that $\sigma_{xx} = 0.003 + 0.25i$ mS and $\sigma_{zz} = 0.03 - 0.76i$ mS, leading to the desired beam. Again, excellent agreement is found between the desired and final beam angles.

As a final example for the graphene strip array, Fig. 10(c) shows $E_y$ for the case of multilayer graphene strips (to increase the conductivity). By using five layers of graphene with $\mu_c = 1$ eV,
tive, so that the surface is not able to support TE modes (the

Using (28), (29), and (31), a surface with these conductivity

and $q_z$ plane. One important difference between branch cuts in
this case and in the previous hyperbolic cases is the branch
cut trajectory. From (35) for the hyperbolic case, because of
the condition $\text{Im}(\sigma_{xx}) \text{Im}(\sigma_{zz}) < 0$, the branch cut trajectory
was along the real axis, but for the anisotropic nonhyperbolic
case, we have $\text{Im}(\sigma_{xx}) \text{Im}(\sigma_{zz}) > 0$ and so the trajectory for
large $q_z$ is parallel to the imaginary axis.

As shown in Fig. 11(b), this anisotropic nonhyperbolic
surface can support a directed SPP, although the beam is
directed primarily along one of the coordinate axes. The
electric field computed as a real-line integral (32) is in good
agreement with the electric field obtained as a sum of branch
cut integrals (33). Fig. 11(c) shows the SPP field in the
logarithmic scale calculated by numerically solving Maxwell’s
equations using a commercial finite-difference time-domain
method (FDTD) from Numerical solutions [45]. Good agree-
ment with the results obtained by complex plane analysis is
observed. Fig. 11(d) shows the vertical variation of the beam in
the logarithmic scale calculated by Numerical, showing strong
SPP confinement to the surface. Using Green’s function the
attenuation length was found to be $p = \lambda/12\pi$.

VII. Conclusion

We have studied the electromagnetic response of
2-D anisotropic and hyperbolic surfaces and developed a
method (based on complex plane analysis) for the efficient
computation of electric field excited on such surfaces.
A solution in term of electric field Sommerfeld integrals
has been obtained for the electromagnetic field due to a
vertical dipole current source located in close proximity
to the surface. Poles, branch points, and related branch
cuts and their relative importance and physical meaning for
surface wave propagation have been emphasized. A first-order
approximation has also been obtained using the SP method.
Examples have been shown for a graphene strip array and BP.

Appendix A

Graphene Strip Hyperbolic Metasurface

A schematic of an array of graphene strips is shown in
Fig. 12(a). This densely packed strip surface can act as
a physical implementation of a metasurface at terahertz and
near-infrared frequencies [3], [46]. The dispersion topology of
the proposed structure may range from elliptical to hyperbolic
as a function of its geometrical and electrical parameters.
The in-plane effective conductivity tensor of the proposed
structure can be analytically obtained using an effective
medium theory as [3]

$$\sigma_{xx}^\text{eff} = \sigma W/L \quad \text{and} \quad \sigma_{zz}^\text{eff} = \frac{L\sigma\sigma_c}{W\sigma_c + G\sigma} \tag{41}$$

where $L$ and $W$ are the periodicity and width of the strips,
respectively, $G = L - W$ is the separation distance between
two consecutive strips, $\sigma$ is graphene conductivity (36),
and $\sigma_c = f(\omega\epsilon_0 L/\pi)\ln(csc(\pi G/2L))$ is an equivalent
conductivity associated with the near-field coupling between
adjacent strips obtained using an electrostatic approach [47].
These effective parameters are valid only when the homogeneity condition $L \ll \lambda_{\text{SPP}}$ is satisfied, where $\lambda_{\text{SPP}}$
BP is an anisotropic monolayer or thin-film material that can support surface plasmons [50]. Fig. 13 shows the in-plane conductivity tensor components at two doping levels, 10 × 10^{13}/cm^2 and 5 × 10^{12}/cm^2, obtained from a Kubo formula as described in [23]. For a 10-nm BP film, the electronic bandgap is approximately 0.5 eV. This accounts for the observed interband absorption along the x polarization, and also characterized by weak interband absorption along z.

It can be seen that by increasing the doping level, larger conductivity components are attainable but the hyperbolic region is also pushed toward higher frequencies. In Fig. 13(a) and (b), BP is an inductive anisotropic (nonhyperbolic) surface, while in Fig. 13(c) and (d), regions 1 and 3 show anisotropic inductive and capacitive responses, respectively, and region 2 shows the anisotropic hyperbolic region.

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REFERENCES


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