Dyadic Eigenfunctions and Natural Modes for Hybrid Waves in Planar Media

George W. Hanson, *Senior Member, IEEE*

***Abstract*—A new formulation for studying electromagnetic wave propagation in an open, planarly layered medium is presented based on eigenfunctions of the Hertzian potential dyadic Green's function operator. Due to the complicated coupling of scalar components of potential at material interfaces, elevation of the usual vector eigenfunction problem to dyadic level is found to lead to a convenient, compact representation of wave propagation phenomena. Although we study the source-free problem here, the three columns of the eigenfunction dyadics represent (up to an excitation-dependent amplitude) the vector fields excited by a given three-dimensional source. The general theory of dyadic eigenfunctions is presented, including orthogonality and the possibility of associated functions (root functions) at modal degeneracies, and an example of propagation in a grounded dielectric slab environment is provided.**

***Index Terms*—Dyadics, eigenfunctions, layered media, natural modes.**

I. Introduction

THE PROPAGATION of electromagnetic waves in a layered medium is typically analyzed by forming differential equations. To examine modal properties of the structure, eigenvalues and eigenfunctions of the resulting differential operators are sometimes studied, although more often one considers the natural modes of the structure, which are the homogeneous solutions of the governing differential equations. The natural modes can also be considered as eigenfunctions corresponding to eigenvalues $\lambda = 0$, at least in a generalized sense.

For a planarly layered medium it is well known that, relative to a certain coordinate, TE, TM, and hybrid modes may exist [1]–[4], depending on the excitation. Thus, there are different types of modes that one is interested in for the source-free problem, physically related to the observed fields arising from the excitation problem. For instance, consider a layered medium vertically stratified along the x-coordinate, analyzed using electric ($\boldsymbol{\pi}_e$) and magnetic ($\boldsymbol{\pi}_m$) Hertzian potentials. To examine the modes of the structure, one typically considers a two-dimensional problem, invariant in one of the longitudinal coordinates (say, the y-coordinate), with propagation $e^{-j\beta z}$. The assumption $\boldsymbol{\pi}_e = \hat{\mathbf{x}}\, \pi_{ex}$, $\boldsymbol{\pi}_m = \mathbf{0}$ leads to TM^z modes, whereas $\boldsymbol{\pi}_e = \hat{\mathbf{y}}\, \pi_{ey}$, $\boldsymbol{\pi}_m = \mathbf{0}$ leads to TE^z modes. The propagation constant β satisfies what we will call the corresponding TE and TM transcendental equations $z^{\mathrm{te}}(\beta) = 0$ and $z^{\mathrm{tm}}(\beta) = 0$, respectively. Alternatively, longitudinal-section electric (LSE, $\boldsymbol{\pi}_e = \mathbf{0}$, $\boldsymbol{\pi}_m = \hat{\mathbf{x}}\, \pi_{mx}$) and magnetic (LSM, $\boldsymbol{\pi}_e = \hat{\mathbf{x}}\, \pi_{ex}$, $\boldsymbol{\pi}_m = \mathbf{0}$) modes may be considered [4, Ch. 6], again leading to independent scalar modes of the waveguide. In this case, one will find that for LSE (LSM) modes β satisfies the aforementioned TE (TM) transcendental equation. If desired, the Green's function or field solution due to a line source excitation can be constructed from a linear superposition of these uncoupled modes [3], [4].

While this decoupling greatly simplifies modal analysis, in the general case a three-dimensional (3-D) source cannot individually excite these two-dimensional modes (TE, TM, LSE, or LSM). Generalizing to allow for arbitrary variation in the longitudinal (y, z) coordinates, one finds that individual scalar components of Hertzian potential become coupled upon enforcing field continuity at planar dielectric interfaces [5]. For example, in a planarly layered medium (horizontal layers), a horizontal electric dipole will lead to both vertical and horizontal components of electric Hertzian potential, which are coupled together [6], although a vertical electric dipole will maintain only a vertical component of electric Hertzian potential. Alternatively, one obtains coupled LSE and LSM modes. These coupled equations are often associated with consideration of the source-driven problem rather than the natural mode (source-free) problem. It is in the sense of coupled horizontal and vertical potentials that we use the term "hybrid" here.

In this paper a physically insightful source-free modal method is developed based on eigenfunctions of the differential operator for the Hertzian potential dyadic Green's function, leading to dyadic eigenfunctions and dyadic natural modes. In general, the dyadic eigenfunctions and natural modes have coupled scalar components, and as special cases decouple to result in the familiar two-dimensional TE and TM or LSE and LSM modes of the structure. This formulation essentially represents an elevation of the classic Sommerfeld problem, in the source-free case, to dyadic level. Columns of the dyadic eigenfunctions can be considered as 3-D vector modes of the structure, which are associated with the response of a point source. In the general case, the natural relationship between hybrid field components is obtained via the corresponding relationship between scalar components of the dyadic mode. An attractive feature of the developed theory is that it represents a straightforward extension of the classical scalar theory of two-dimensional eigenfunctions of a layered medium via Sturm–Liouville analysis (see, e.g., [3], [4], and [7]) to vector electromagnetic phenomena. The general theory of dyadic eigenfunctions is presented, dyadic adjoint modes and dyadic

Manuscript received April 9, 2003.
The author is with the Department of Electrical Engineering, University of Wisconsin-Milwaukee, Milwaukee, WI 53211 USA (e-mail: george@uwm.edu).
Digital Object Identifier 10.1109/TAP.2004.825673

0018-926X/04$20.00 © 2004 IEEE

associated functions are discussed, and orthogonality properties of the dyadic modes are obtained.

Although dyadic eigenfunctions were originally introduced in [8], in that work we considered a source-driven vertically closed structure, and the emphasis was on interpreting residue contributions of the spectral (Sommerfeld form) dyadic Green's function in the event of modal degeneracies. Dyadic eigenfunctions of the vertical-component Green's function operator (i.e., the operator obtained subsequent to Fourier transformation on longitudinal coordinates) were introduced to interpret the residue contributions as eigenfunctions and associated functions. In this paper we consider an open geometry and focus on general properties of the dyadic eigenfunctions and dyadic associated functions of the 3-D spatial Green's function operator for the source-free propagation problem.

II. Formulation

Consider a planarly layered medium inhomogeneous in the x-coordinate, forming M material layers, with the jth layer being

$$\Omega_j = \{(x,y,z) : x \in (a_{j-1}, a_j), -\infty < y, z < \infty\}, \quad j = 1, \ldots, M. \quad (1)$$

The lower upper (lower) interface a_0 (a_M) can reside at $-\infty$ ($+\infty$) to implement a semi-infinite or infinite region. Assume $\varepsilon(x)$ is a piecewise constant, complex-valued function, where $\varepsilon(x) = \varepsilon_j$ for $x \in (a_{j-1}, a_j)$ with $\mathrm{Im}\{\varepsilon_j\} < 0$. As an example, a two-layer structure typical of microstrip antenna environments is shown in Fig. 1.

The Hertzian potential Green's dyadic $\underline{\mathbf{g}}(\mathbf{r}, \mathbf{r}')$ provides the field at $\mathbf{r}$ due to an elemental current source at $\mathbf{r}' \in \Omega_i$ and is not, in general, a diagonal dyadic. The Green's dyadic is a solution of

$$-(\nabla^2 + k^2(x))\underline{\mathbf{g}}(\mathbf{r}, \mathbf{r}') = \underline{\mathbf{I}}\boldsymbol{\delta}(\mathbf{r} - \mathbf{r}') \quad (2)$$

where $k(x) = \omega\sqrt{\mu_0 \varepsilon(x)}$, subject to boundary and continuity conditions imposed on the electric Hertzian potential [9] elevated to dyadic level. At each dielectric interface a_i, $i = 1, \ldots, M-1$

$$g_{x\beta}^{+} = N_{\mp}^{2} g_{x\beta}^{-}, \; g_{\alpha\alpha}^{+} = N_{\mp}^{2} g_{\alpha\alpha}^{-} \quad (3)$$

$$\left(\frac{\partial g_{xx}^{+}}{\partial x} - \frac{\partial g_{xx}^{-}}{\partial x}\right) = 0, \; \frac{\partial g_{\alpha\alpha}^{+}}{\partial x} = N_{\mp}^{2} \frac{\partial g_{\alpha\alpha}^{-}}{\partial x} \quad (4)$$

$$\left(\frac{\partial g_{x\alpha}^{+}}{\partial x} - \frac{\partial g_{x\alpha}^{-}}{\partial x}\right) = -(N_{\mp}^{2} - 1)\frac{\partial g_{\alpha\alpha}^{-}}{\partial \alpha}, \quad \beta = x, y, z, \quad \alpha = y, z \quad (5)$$

where $N_{\mp}^{2} = \varepsilon^{-}/\varepsilon^{+}$ with $\varepsilon^{-(+)}$ being the permittivity just below (above) the interface. At any perfectly conducting walls (possibly a_0 and/or a_M) boundary conditions are

$$g_{\alpha\alpha} = 0, \qquad \alpha = y, z \quad (6)$$

$$\frac{\partial g_{x\beta}}{\partial x} = 0, \qquad \beta = x, y, z. \quad (7)$$

Assuming lossy medium, the Green's dyadic must also vanish at infinity.

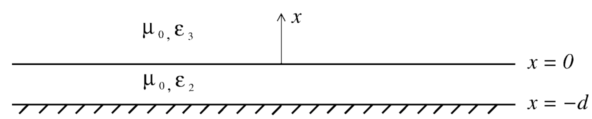

Fig. 1. Grounded dielectric slab waveguide.

Since we are interested in the eigenvalue and natural mode problem, we consider the homogeneous form of (2) in what follows. Define an inner product as

$$\langle \underline{\mathbf{u}}, \underline{\mathbf{v}} \rangle = \int_{\Omega} \underline{\mathbf{u}}(\mathbf{r}) \colon \overline{\underline{\mathbf{v}}}(\mathbf{r}) d\Omega = \int_{\Omega} \sum_{i,j=1}^{3} u_{ij}(\mathbf{r}) \overline{v}_{ij}(\mathbf{r}) d\Omega \quad (8)$$

utilizing the double-dot product notation [10], where the overbar indicates complex conjugation and $\Omega = \cup_{j=1}^{M} \Omega_j$. The space $\mathbf{L}^2(\Omega)$ is the space of functions such that

$$\|\underline{\mathbf{u}}\|^2 = \langle \underline{\mathbf{u}}, \underline{\mathbf{u}} \rangle < \infty. \quad (9)$$

The operator of the dyadic Green's function problem L is defined by its formal action and its domain, such that

$$(L\underline{\mathbf{g}})(\mathbf{r}) \equiv -(\nabla^2 + k^2(x))\underline{\mathbf{g}}(\mathbf{r}), \quad D_L \equiv \{\underline{\mathbf{g}} \in \mathbf{L}^2(\Omega) : (3) - (7) \text{hold}\}. \quad (10)$$

A. Dyadic Eigenfunctions

Dyadic eigenfunctions of the operator L, $\underline{\mathbf{u}}_n(\mathbf{r}) \neq \underline{\mathbf{0}}$, which have scalar components $u_{n,ij}$, $i, j = x, y, z$, satisfy

$$-(\nabla^2 + k^2(x))\underline{\mathbf{u}}_n(\mathbf{r}) = \lambda_n \underline{\mathbf{u}}_n(\mathbf{r}), \quad \underline{\mathbf{u}}_n \in D_L. \quad (11)$$

The eigenequation (11) can be solved via separation of variables, leading to the 3-D eigenfunctions in rectangular coordinates as

$$\underline{\mathbf{u}}_n(\mathbf{r}) = \underline{\mathbf{u}}_{n,k_y,k_z}(\mathbf{r}) = \underline{\mathbf{u}}_n(x) \frac{e^{ik_y y}}{\sqrt{2\pi}} \frac{e^{ik_z z}}{\sqrt{2\pi}} \quad (12)$$

where $k_y, k_z \in (-\infty, \infty)$ are continuous variables. In this case $u_{k_\alpha}(\alpha) = e^{ik_\alpha \alpha}/\sqrt{2\pi}$, $\alpha = y, z$, can be considered as an improper eigenfunction [11], [7], which represents the continuous spectrum of the operator. Improper eigenfunctions are not in the domain of the operator since they do not lead to eigenfunctions that are in $\mathbf{L}^2(\Omega)$. For improper eigenfunctions one merely obtains boundedness at infinity

$$\lim_{|\alpha| \to \infty} |u_{k_\alpha}(\alpha)| = O(1) \quad (13)$$

and therefore in determining improper eigenfunctions we relax the $\mathbf{L}^2$ requirement. However, the integral superposition of the improper eigenfunctions is in $\mathbf{L}^2(\Omega)$, which is related to the case of an inverse (longitudinal) Fourier transform being an $\mathbf{L}^2$ function.

The x-coordinate eigenfunctions $\underline{\mathbf{u}}_n(x)$ satisfy

$$-\left(\frac{d^2}{dx^2} + k^2(x)\right)\underline{\mathbf{u}}_n(x) = \xi_n \underline{\mathbf{u}}_n(x), \quad x \in (a_0, a_M) \quad (14)$$

where $\xi_n = \lambda_n - k_\rho^2$ and $k_\rho^2 = k_y^2 + k_z^2$, leading to the homogeneous equation

$$\left(\frac{d^2}{dx^2} - \gamma^2(x)\right)\underline{\mathbf{u}}_n(x) = 0, \quad x \in (a_0, a_M) \tag{15}$$

where $\gamma^2(x) = -\xi_n - k^2(x) = k_\rho^2 - \lambda_n - k^2(x)$.

Due to the fact that the structure is potentially a vertically open environment (e.g., as depicted in Fig. 1), the dyadic eigenfunctions $\underline{\mathbf{u}}_n(x)$ belong to two classes. Proper eigenfunctions vanish at vertical infinity, leading to TE and TM transcendental equations $z^{\text{te}}(\xi_n) = 0$ and $z^{\text{tm}}(\xi_n) = 0$, respectively, and to the value of ξ_n. The eigenvalue (in a general sense, allowing for the longitudinal continuous spectrum) is $\lambda_n = \xi_n + k_\rho^2$, where $0 \le k_\rho < \infty$. Eigenfunctions improper in the vertical coordinate (related to radiation modes) are merely bounded at vertical infinity

$$\lim_{|x|\to\infty} |u_{n,ij}(x)| = O(1), \quad i,j = x,y,z \tag{16}$$

where $n = \upsilon$ is a continuous index and ξ_υ is a continuous variable, where the eigenvalue $\lambda_n = \xi_n + k_\rho^2$ is purely continuous. In the latter case the improper eigenfunction will be denoted as $\underline{\mathbf{u}}_{\upsilon,k_y,k_z}(\mathbf{r})$, although the index n will be used for the general development (representing both discrete and continuous variables). Note, that $\underline{\mathbf{u}}_n(x) = \underline{\mathbf{u}}_n(x, \xi_n)$.

Alternative to (12), in cylindrical coordinates the eigenfunctions are

$$\underline{\mathbf{u}}_n(\mathbf{r}) = \underline{\mathbf{u}}_n(x, \xi_n)\frac{e^{im\phi}}{\sqrt{2\pi}} J_m(k_\rho \rho) \tag{17}$$

where $\underline{\mathbf{u}}_n(x)$ satisfies (15), leading to the value of ξ_n, and where $0 \le k_\rho < \infty$ and $m = \pm 1, \pm 2, \pm 3, \ldots$. In either coordinate system natural modes of the structure are obtained by setting $\lambda_n = 0$. For instance, in cylindrical coordinates we obtain natural modes as

$$\underline{\mathbf{u}}_n(x, \xi_n)\frac{e^{im\phi}}{\sqrt{2\pi}} J_m(\sqrt{-\xi_n}\rho). \tag{18}$$

From (3)–(7) it is clear that only dyadic components $u_{n,x\beta}$, $\beta = x,y,z$, and $u_{n,\alpha\alpha}$, $\alpha = y,z$, are nonzero, and, moreover, that the equations for $u_{n,xx}$ and for the pairs $(u_{n,\alpha\alpha}(x), u_{n,x\alpha}(x))$, $\alpha = y,z$, all decouple from each other. Considering that the propagation constant for proper modes may satisfy either the TM ($z^{\text{tm}} = 0$) or TE ($z^{\text{te}} = 0$) transcendental equations, and that $u_{n,xx}$ will always be associated with TM eigenvalues, we obtain five decoupled (independent) dyadic problems

$$-\left(\frac{d^2}{dx^2} + k^2(x)\right)\underline{\mathbf{u}}_{n_e}^{xx}(x) = \xi_{n_e}\underline{\mathbf{u}}_{n_e}^{xx}(x) \tag{19}$$

$$-\left(\frac{d^2}{dx^2} + k^2(x)\right)\underline{\mathbf{u}}_{n_e}^{\alpha\alpha}(x) = \xi_{n_e}\underline{\mathbf{u}}_{n_e}^{\alpha\alpha}(x), \ \alpha = y,z \tag{20}$$

$$-\left(\frac{d^2}{dx^2} + k^2(x)\right)\underline{\mathbf{u}}_{n_h}^{\alpha\alpha}(x) = \xi_{n_h}\underline{\mathbf{u}}_{n_h}^{\alpha\alpha}(x), \ \alpha = y,z \tag{21}$$

with n_e indicating TM (E) modes and n_h indicating TE (H) modes, where

$$\underline{\mathbf{u}}_{n_e}^{xx}(x) = \begin{bmatrix} u_{n_e,xx}(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\underline{\mathbf{u}}_n^{yy}(x) = \begin{bmatrix} 0 & u_{n,xy}(x) & 0 \\ 0 & u_{n,yy}(x) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\underline{\mathbf{u}}_n^{zz}(x) = \begin{bmatrix} 0 & 0 & u_{n,xz}(x) \\ 0 & 0 & 0 \\ 0 & 0 & u_{n,zz}(x) \end{bmatrix} \tag{22}$$

and n without a subscript can denote either n_e or n_h. Note, that $\underline{\mathbf{u}}_n^{\alpha\alpha}$, $\alpha = y,z$, needs only to be obtained for one α, either $\alpha = y$ or $\alpha = z$; up to an arbitrary constant the corresponding nonzero entries for the other value of α have the same form.

To discuss orthogonality, since the operator L is non-self-adjoint we must consider dyadic eigenfunctions adjoint to the ones considered above. With the inner product (8), from the definition of the adjoint operator L^*

$$\langle L\underline{\mathbf{u}}, \underline{\mathbf{v}}\rangle = \langle \underline{\mathbf{u}}, L^*\underline{\mathbf{v}}\rangle \tag{23}$$

the operator adjoint to (10) is

$$\begin{aligned}(L^*\underline{\mathbf{v}})(\mathbf{r}) &\equiv -(\nabla^2 + \overline{k}^2(x))\underline{\mathbf{v}}(\mathbf{r}) \\ D_{L^*} &\equiv \{\underline{\mathbf{v}} \in \mathbf{L}^2(\Omega) : (25)-(30) \text{ hold}\}.\end{aligned} \tag{24}$$

The adjoint continuity conditions (25)–(28), which are quite different than (3)–(5), are obtained as

$$v_{n,x\beta}^- = v_{n,x\beta}^+, \ v_{n,\alpha\alpha}^- = \overline{N_\mp}^2 v_{n,\alpha\alpha}^+ \tag{25}$$

$$\frac{\partial v_{n,x\beta}^-}{\partial x} = \overline{N_\mp}^2 \frac{\partial v_{n,x\beta}^+}{\partial x} \tag{26}$$

$$\frac{\partial v_{n,\alpha\alpha}^-}{\partial x} - \overline{N_\mp}^2 \frac{\partial v_{n,\alpha\alpha}^+}{\partial x} = -(\overline{N_\mp}^2 - 1)\frac{\partial v_{n,x\alpha}^+}{\partial \alpha} \tag{27}$$

$$\beta = x,y,z, \quad \alpha = y,z \tag{28}$$

at the dielectric interfaces a_i, $i = 1, \ldots, M-1$. Adjoint boundary conditions at any perfectly conducting walls are the same as (6)–(7)

$$v_{n,\alpha\alpha} = 0, \quad \alpha = y,z \tag{29}$$

$$\frac{\partial v_{n,x\beta}}{\partial x} = 0, \quad \beta = x,y,z \tag{30}$$

leading to the adjoint eigenvalue problem

$$-(\nabla^2 + \overline{k}^2(x))\underline{\mathbf{v}}_n(\mathbf{r}) = \lambda_n^* \underline{\mathbf{v}}_n(\mathbf{r}), \quad \underline{\mathbf{v}}_n \in D_{L^*} \tag{31}$$

where $\lambda_n^* = \overline{\lambda_n}$, with adjoint eigenfunctions

$$\underline{\mathbf{v}}_n(\mathbf{r}) = \underline{\mathbf{v}}_{n,k_y,k_z}(\mathbf{r}) = \underline{\mathbf{v}}_n(x)\frac{e^{ik_y y}}{\sqrt{2\pi}}\frac{e^{ik_z z}}{\sqrt{2\pi}} \tag{32}$$

where $k_y, k_z \in (-\infty, \infty)$ are continuous variables. As before, improper adjoint eigenfunctions are not in $\mathbf{L}^2$.

The vertical component functions $\underline{\mathbf{v}}_n(xt)$ satisfy the five eigenvalue problems

$$-\left(\frac{d^2}{dx^2}+\overline{k}^2(x)\right)\underline{\mathbf{v}}_{n_e}^{xx}(x)=\xi_{n_e}^*\underline{\mathbf{v}}_{n_e}^{xx}(x) \quad (33)$$

$$-\left(\frac{d^2}{dx^2}+\overline{k}^2(x)\right)\underline{\mathbf{v}}_{n_e}^{\alpha\alpha}(x)=\xi_{n_e}^*\underline{\mathbf{v}}_{n_e}^{\alpha\alpha}(x),\ \alpha=y,z \quad (34)$$

$$-\left(\frac{d^2}{dx^2}+\overline{k}^2(x)\right)\underline{\mathbf{v}}_{n_h}^{\alpha\alpha}(x)=\xi_{n_h}^*\underline{\mathbf{v}}_{n_h}^{\alpha\alpha}(x),\ \alpha=y,z \quad (35)$$

where $\xi_n^*=\overline{\xi}_n$. From the boundary and continuity conditions it is found that the vertical coordinate adjoint eigenfunctions have the relatively simple forms

$$\underline{\mathbf{v}}_{n_e}^{xx}(x)=\begin{bmatrix} v_{n_e,xx}(x) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$
$$\underline{\mathbf{v}}_{n}^{yy}(x)=\begin{bmatrix} 0 & v_{n,xy}(x) & 0\\ 0 & v_{n,yy}(x) & 0\\ 0 & 0 & 0\end{bmatrix}$$
$$\underline{\mathbf{v}}_{n}^{zz}(x)=\begin{bmatrix} 0 & 0 & v_{n,xz}(x)\\ 0 & 0 & 0\\ 0 & 0 & v_{n,zz}(x)\end{bmatrix}. \quad (36)$$

With a slight abuse of notation, let n represent the triplet (n,k_y,k_z) and m the triplet (m,k_y',k_z') It is a simple matter to obtain the orthogonality relationship

$$(\lambda_n-\lambda_m)\langle\underline{\mathbf{u}}_n^{\beta\beta},\underline{\mathbf{v}}_m^{\beta\beta}\rangle=0,\quad \beta=x,y,z. \quad (37)$$

Furthermore, by properties of the dyadic double-dot product

$$\langle\underline{\mathbf{u}}_n^{\alpha\alpha},\underline{\mathbf{v}}_m^{\beta\beta}\rangle=0,\quad \alpha,\beta=x,y,z,\quad \alpha\neq\beta. \quad (38)$$

Eigenfunctions proper in the vertical coordinate are normalized as

$$\langle\underline{\mathbf{u}}_n^{\beta\beta},\underline{\mathbf{v}}_m^{\beta\beta}\rangle=\delta(k_y-k_y')\delta(k_z-k_z')\delta_{nm},\quad \beta=x,y,z \quad (39)$$

where $\delta(v-v)$ is the Dirac delta function and δ_{nm} is the Kronecker delta function. When $n=v$ represents the continuous triplet (v,k_y,k_z) and $m=v'$ the continuous triplet (v',k_y',k_z'), eigenfunctions improper in the vertical coordinate are normalized as

$$\langle\underline{\mathbf{u}}_n^{\beta\beta},\underline{\mathbf{v}}_m^{\beta\beta}\rangle=\delta(k_y-k_y')\delta(k_z-k_z')\delta(v-v'),\quad \beta=x,y,z. \quad (40)$$

B. Dyadic Root Functions

Since the operator L is not self-adjoint, eigenvalues of multiplicity $q>1$ may occur. Since $\lambda_n=\xi_n+k_\rho^2$, this will happen when

$$z(\xi)|_{\xi_n}=\frac{d}{d\xi}z(\xi)\bigg|_{\xi_n}=\cdots=\frac{d^{q-1}}{d\xi^{q-1}}z(\xi)\bigg|_{\xi_n}=0$$

where z is either $z^{\rm te}$ or $z^{\rm tm}$. In this case, eigenfunctions generally will not form a basis for the space in question (an exception would be the case when q linearly independent eigenfunctions exist, which would form a trivial degeneracy). In the case of nontrivial model degeneracies, in addition to eigenfunctions, it is necessary to consider root functions [12]–[15], [8], similar to the idea of a Jordan chain in matrix theory.

Consider L as defined by (10). An element $\underline{\mathbf{0}}\neq\underline{\mathbf{u}}_{n,q-1}$ is a root function of rank q of the operator L corresponding to an eigenvalue λ_n if

$$(L-\lambda_nI)^q\underline{\mathbf{u}}_{n,q-1}=\underline{\mathbf{0}}$$
$$(L-\lambda_nI)^{q-1}\underline{\mathbf{u}}_{n,q-1}\neq\underline{\mathbf{0}} \quad (41)$$

where q is a positive integer. Every eigenfunction of L is a root function of rank 1 ($\underline{\mathbf{u}}_{n,0}\equiv\underline{\mathbf{u}}_n$), and the root functions having rank $q>1$ are called associated functions (functions associated with the eigenfunction). The root system of L is defined as the union of the eigenfunctions and the associated functions.

In practice, to determine the associated functions one starts with an eigenfunction $\underline{\mathbf{u}}_n=\underline{\mathbf{u}}_{n,0}$ satisfying $(L-\lambda_nI)\,\underline{\mathbf{u}}_n=\underline{\mathbf{0}}$. If the equation

$$(L-\lambda_nI)\underline{\mathbf{u}}_{n,1}=\underline{\mathbf{u}}_n \quad (42)$$

has a solution $\underline{\mathbf{u}}_{n,1}$, then $\underline{\mathbf{u}}_{n,1}$ is a root function of rank 2; more specifically, an associated function associated with the eigenvalue λ_n and eigenfunction $\underline{\mathbf{u}}_n$. In general, we consider

$$(L-\lambda_nI)\underline{\mathbf{u}}_{n,k}=\underline{\mathbf{u}}_{n,k-1} \quad (43)$$

such that the chain $\{\underline{\mathbf{u}}_n,\underline{\mathbf{u}}_{n,1},\underline{\mathbf{u}}_{n,2},\ldots\underline{\mathbf{u}}_{n,j}\}$ consisting of the eigenfunctions and associated functions is called a Jordan or Keldysh chain of length $j+1$ corresponding to the eigenvalue λ_n. The same ideas apply to the adjoint eigenfunctions.

It is straightforward to obtain the orthogonality relationships

$$(\lambda_n-\lambda_m)\langle\underline{\mathbf{u}}_{n,p}^{\beta\beta},\underline{\mathbf{v}}_{m,q}^{\beta\beta}\rangle+\langle\underline{\mathbf{u}}_{n,p-1}^{\beta\beta},\underline{\mathbf{v}}_{m,q}^{\beta\beta}\rangle-\langle\underline{\mathbf{u}}_{n,p}^{\beta\beta},\underline{\mathbf{v}}_{m,q-1}^{\beta\beta}\rangle=0 \quad (44)$$

for $\beta=x,y,z$ and $p,q=0,1,2,\ldots$, where for notational convenience we define $\underline{\mathbf{u}}_{n,p}^{\beta\beta}\equiv 0$ for $p<0$, and that

$$\langle\underline{\mathbf{u}}_{n,p}^{\alpha\alpha},\underline{\mathbf{v}}_{m,q}^{\beta\beta}\rangle=0,\ \alpha,\beta=x,y,z,\ \alpha\neq\beta,\ p,q=0,1,2,\ldots. \quad (45)$$

Note, that if $\lambda_n\neq\lambda_m$, then recursively (starting with $p=q=0$) we see that

$$\langle\underline{\mathbf{u}}_{n,p}^{\beta\beta},\underline{\mathbf{v}}_{m,q}^{\beta\beta}\rangle=0,\quad p,q=0,1,2,\ldots. \quad (46)$$

In particular, if only the kth eigenvalue has multiplicity of two (all others having unit multiplicity), we consider the set $\{\underline{\mathbf{u}}_k^{\beta\beta},\underline{\mathbf{u}}_{k,1}^{\beta\beta},\underline{\mathbf{v}}_k^{\beta\beta},\underline{\mathbf{v}}_{k,1}^{\beta\beta}\}$ corresponding to the double eigenvalue λ_k and $\{\underline{\mathbf{u}}_n^{\beta\beta},\underline{\mathbf{v}}_n^{\beta\beta}\}$ corresponding to the other eigenvalues $n\neq k$. Then

$$\langle\underline{\mathbf{u}}_n^{\beta\beta},\underline{\mathbf{v}}_m^{\beta\beta}\rangle=0,\quad n\neq m,\quad n,m\neq k \quad (47)$$
$$\langle\underline{\mathbf{u}}_n^{\beta\beta},\underline{\mathbf{v}}_k^{\beta\beta}\rangle=\langle\underline{\mathbf{u}}_n^{\beta\beta},\underline{\mathbf{v}}_{k,1}^{\beta\beta}\rangle=0,\quad n\neq k \quad (48)$$
$$\langle\underline{\mathbf{u}}_k^{\beta\beta},\underline{\mathbf{v}}_k^{\beta\beta}\rangle=0 \quad (49)$$
$$\langle\underline{\mathbf{u}}_k^{\beta\beta},\underline{\mathbf{v}}_{k,1}^{\beta\beta}\rangle=\langle\underline{\mathbf{u}}_{k,1}^{\beta\beta},\underline{\mathbf{v}}_k^{\beta\beta}\rangle. \quad (50)$$

The third condition $\langle\underline{\mathbf{u}}_k^{\beta\beta},\underline{\mathbf{v}}_k^{\beta\beta}\rangle=0$ is quite different than in the case of rank 1 root functions (eigenfunctions), in which case, regardless of normalization $\langle\underline{\mathbf{u}}_k^{\beta\beta},\underline{\mathbf{v}}_k^{\beta\beta}\rangle\neq 0$. Thus, points of model degeneracy require special treatment and consideration of associated functions as described above. In the source-driven problem using a Sommerfeld-type representation of the dyadic Green's function, at points of model degeneracy one obtains poles of order $q > 1$, and corresponding second-order residues yield the associated function contribution [8]. Therefore, one can see that associated functions are physically meaningful in the case of nontrivial model degeneracies.

By taking derivatives of (43) with respect to the spectral variable λ, it can be shown that

$$\underline{\mathbf{u}}_{n,p}^{\beta\beta}(\mathbf{r},\lambda_n) = \left(\frac{1}{p!}\frac{\partial^p}{\partial\lambda^p} - c\right)\underline{\mathbf{u}}_n^{\beta\beta}(\mathbf{r},\lambda)\bigg|_{\lambda=\lambda_n} \tag{51}$$

where c is an arbitrary constant. Therefore, associated functions are related to derivatives of eigenfunctions and involve factors that grow spatially [see, e.g., (86)–(88)].

Finally, we consider the meaning of the dyadic eigenfunctions and natural modes. The electric and magnetic fields due to an electric Hertzian potential are

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= (k^2(x) + \nabla\nabla\cdot)\boldsymbol{\pi}_e(\mathbf{r})\\ \mathbf{H}(\mathbf{r}) &= i\omega\varepsilon(x)\nabla\times\boldsymbol{\pi}_e(\mathbf{r})\end{aligned} \tag{52}$$

$\mathbf{r} \in \Omega$. Since (2) is the equation for the Hertzian potential Green's dyadic and (10) is the operator associated with (2), the dyadic eigenfunctions and natural modes represent the source-free Hertzian potential elevated to dyadic level. Thus, considering the general dyadic

$$\underline{\mathbf{u}}_n(x) = \begin{bmatrix} u_{n,xx}(x) & u_{n,xy}(x) & u_{n,xz}(x)\\ 0 & u_{n,yy}(x) & 0\\ 0 & 0 & u_{n,zz}(x)\end{bmatrix} \tag{53}$$

we interpret the first column as the vector Hertzian potential due to a vertical (x) component of electric current. The second and third columns are the vector Hertzian potential due to y– and z-components of electric current, respectively. To obtain electric and magnetic fields (either relating to eigenfunctions of potential or natural-mode potentials), one can form dyadic fields

$$\begin{aligned}\underline{\mathbf{E}}(\mathbf{r}) &= (k^2(x) + \nabla\nabla\cdot)\underline{\mathbf{u}}_n(\mathbf{r})\\ \underline{\mathbf{H}}(\mathbf{r}) &= i\omega\varepsilon(x)\nabla\times\underline{\mathbf{u}}_n(\mathbf{r})\end{aligned} \tag{54}$$

where the resulting columns of $\underline{\mathbf{E}}$, $\underline{\mathbf{H}}$ have the same meaning as described above for the dyadic potentials. The dyadic Green's function for Hertzian potential can be formed by summing over the dyadic eigenfunctions using the appropriate scalar excitation amplitudes, but that problem will be considered elsewhere.

III. Example

As an example, consider the grounded slab waveguide typical of printed antenna structures as shown in Fig. 1. Dyadic eigenfunctions discrete in the vertical coordinate are obtained from (11) subject to the condition that they vanish at vertical infinity, leading to the proper surface-wave modes. Let

$$n^{(3)}(x) \equiv e^{-\gamma_3 x} \tag{55}$$

$$\begin{aligned}n_1^{(2)}(x) &\equiv \{\cosh(\gamma_2 d)\cosh(\gamma_2 x) + \sinh(\gamma_2 d)\sinh(\gamma_2 x)\}\\ &= \cosh(\gamma_2(d+x))\end{aligned} \tag{56}$$

$$\begin{aligned}n_2^{(2)}(x) &\equiv \{\sinh(\gamma_2 d)\cosh(\gamma_2 x) + \cosh(\gamma_2 d)\sinh(\gamma_2 x)\}\\ &= \sinh(\gamma_2(d+x))\end{aligned} \tag{57}$$

where $\gamma_j^2 = -\xi_n - k_j^2$, $j = 2, 3$, with the well-known transcendental equations for TE and TM modes of the slab

$$z^{\rm tm}(\xi_n) = N^2\gamma_3\cosh(\gamma_2 d) + \gamma_2\sinh(\gamma_2 d) \tag{58}$$

$$z^{\rm te}(\xi_n) = \gamma_2\cosh(\gamma_2 d) + \gamma_3\sinh(\gamma_2 d) \tag{59}$$

where $N^2 = \varepsilon_2/\varepsilon_3$. The eigenfunctions are obtained as (12) where $\underline{\mathbf{u}}_n(x)$ is determined from (15) and presented below.

Two cases arise. If $z^{\rm te}(\xi_n^{\rm te}) = 0$ and $z^{\rm tm}(\xi_n^{\rm te}) \neq 0$, then

$$u_{n,xx}^{(3)}(x) = u_{n,xx}^{(2)}(x) = 0 \tag{60}$$

$$u_{x\alpha}^{(3)}(x) = A_1\left[\frac{N^2(N^2-1)\cosh(\gamma_2 d)}{z^{\rm tm}(\xi_n^{\rm te})}(ik_\alpha)\right]n^{(3)}(x) \tag{61}$$

$$u_{\alpha\alpha}^{(3)}(x) = A_1[N^2]n^{(3)}(x) \tag{62}$$

$$u_{x\alpha}^{(2)}(x) = A_1\left[\frac{(N^2-1)}{z^{\rm tm}(\xi_n^{\rm te})}(ik_\alpha)\right]n_1^{(2)}(x) \tag{63}$$

$$u_{\alpha\alpha}^{(2)}(x) = A_1\left[\frac{1}{\sinh(\gamma_2 d)}\right]n_2^{(2)}(x) \tag{64}$$

for $\alpha = y, z$, where the superscript indicates the region (2 or 3), $\xi_n^{\rm te}$ is the nth root of the TE transcendental equation $z^{\rm te}(\xi_n^{\rm te}) = 0$, $z^{\rm tm}(\xi_n^{\rm te})$ indicates the TM eigenvalue equation evaluated at the TE eigenvalue, and throughout this example $A_i = A_i(\xi_n)$ represent constants with respect to x, y, z.

For the adjoint proper eigenfunctions, we obtain

$$\nu_{n,xx}^{(3)}(x) = \nu_{n,xx}^{(2)}(x) = 0 \tag{65}$$

$$v_{x\alpha}^{(3)}(x) = v_{x\alpha}^{(2)}(x) = 0 \tag{66}$$

$$\overline{v}_{\alpha\alpha}^{(3)}(x) = A_2 n^{(3)}(x) \tag{67}$$

$$\overline{v}_{\alpha\alpha}^{(2)}(x) = A_2\left[\frac{N^2}{\sinh(\gamma_2 d)}\right]n_2^{(2)}(x) \tag{68}$$

with the normalization (39), leading to

$$A_1 A_2 = \frac{1}{N^2}\left(\frac{(1-e^{-4\gamma_2 d}-4\gamma_2 d e^{-2\gamma_2 d})}{8\gamma_2\sinh^2\gamma_2 d}e^2\gamma_2 d + \left(\frac{1}{2\gamma_3}\right)\right)^{-1}.$$

Note, that if there is no variation in the $\alpha = y, z$ direction ($k_\alpha = 0$), then $u_{x\alpha} = 0$. This relates physically to decoupled potentials due to a line-source invariant in the α direction.

This formulation naturally leads to the connection between vertical and horizontal potentials. The ratio of fields at the surface is

$$\begin{aligned}R &= \frac{u_{x\alpha}(0)}{u_{\alpha\alpha}(0)} = \frac{(N^2-1)\cosh(\gamma_2 d)}{z^{\rm tm}(\xi_n^{\rm te})}(ik_\alpha)\\ &= \frac{(N^2-1)\gamma_3}{N^2\gamma_3^2 - \gamma_2^2}(jk_\alpha) = R_0(ik_\alpha).\end{aligned} \tag{69}$$

At low frequencies ($k_0 \ll 1$; $-\xi_n^{\rm te} \neq 0$) we obtain

$$R_0 = \frac{1}{\sqrt{-\xi_n^{\rm te}}} \tag{70}$$

at cutoff ($\gamma_3 = 0$; $-\xi_n^{\rm te} = \varepsilon_{r3}k_0^2$)

$$R_0 = 0 \tag{71}$$

and at high frequencies ($k_0 \gg k_{co}$ where k_{co} is the cutoff wavenumber of the mode; $-\xi_n^{\rm te} \rightarrow \sqrt{\varepsilon_{r2}}k_0$ with ε_{r2} the relative permittivity)

$$R_0 = \frac{\sqrt{\varepsilon_{r2} - \varepsilon_{r3}}}{\varepsilon_{r2}} \frac{1}{k_0}. \tag{72}$$

Therefore, starting at low frequency the leaky modes (below cutoff modes) are coupled (hybrid) modes, and as frequency increases the modes decouple exactly at cutoff (a well-known result). As frequency increases beyond cutoff the modes become coupled again, and at high frequencies $u_{\alpha\alpha}$ becomes (algebraically) large compared to $u_{x\alpha}$. As a numerical example, $|R_0|$ as a function of normalized frequency is plotted in Fig. 2 for a grounded slab as shown in Fig. 1 with $\varepsilon_3 = \varepsilon_0$, $\varepsilon_2 = 2.56\varepsilon_0$. Results for TE$_n$ modes, $n = 1, 3, 5$, and 7 are provided. The coupling and decoupling of potential components is clearly shown (the cutoff wavenumbers are $k_0 d = 1.25, 6.287, 11.32$, and 16.35 for the modes $n = 1, 3, 5, 7$, respectively). Note, that the leaky modes are shown here for reference only. They are not part of the proper spectrum of the operator and are not used in an exact spectral representation.

The second case arises from $z^{\rm tm}(\xi_n^{\rm tm}) = 0$ and $z^{\rm te}(\xi_n^{\rm tm}) \neq 0$, leading to

$$u_{n,xx}^{(3)}(x) = A_5 n^{(3)}(x) \tag{73}$$

$$u_{n,xx}^{(2)}(x) = A_5 \frac{1}{N^2 \cosh(\gamma_2 d)} n_1^{(2)}(x) \tag{74}$$

$$u_{\alpha\alpha}^{(3)} = u_{\alpha\alpha}^{(2)} = 0 \tag{75}$$

$$u_{x\alpha}^{(3)}(x) = A_3 [N^2 \cosh(\gamma_2 d)] n^{(3)}(x) \tag{76}$$

$$u_{x\alpha}^{(2)} = A_3 n_1^{(2)}(x) \tag{77}$$

and

$$\overline{\nu}_{n,xx}^{(3)}(x) = A_6 n^{(3)}(x) \tag{78}$$

$$\overline{\nu}_{n,xx}^{(2)}(x) = A_6 \frac{1}{\cosh(\gamma_2 d)} n_1^{(2)}(x) \tag{79}$$

$$\overline{v}_{x\alpha}^{(3)}(x) = A_4\, n^{(3)}(x) \tag{80}$$

$$\overline{v}_{\alpha\alpha}^{(3)}(x) = A_4 \left[\frac{(N^2 - 1)\sinh(\gamma_2 d)}{N^2 z^{\rm te}(\xi_n^{\rm tm})}(ik_\alpha)\right] n^{(3)}(x) \tag{81}$$

$$\overline{v}_{x\alpha}^{(2)}(x) = A_4 \left[\frac{1}{\cosh(\gamma_2 d)}\right] n_1^{(2)}(x) \tag{82}$$

$$\overline{v}_{\alpha\alpha}^{(2)}(x) = A_4 \left[\frac{(N^2 - 1)}{z^{\rm te}(\xi_n^{\rm tm})}(ik_\alpha)\right] n_2^{(2)}(x), \tag{83}$$

where the normalization (39) leads to

$$A_5 A_6 = \left(\frac{-1 + e^{4\gamma_2 d} + 4\gamma_2 d e^{2\gamma_2 d}}{8N^2\gamma_2 \sinh^2\gamma_2 d} e^{-2\gamma_2 d} + \left(\frac{1}{2\gamma_3}\right)\right)^{-1},$$
$$A_3 A_4 = (N^2 \cosh(\gamma_2 d))^{-1} A_5 A_6.$$

As an example of determining associated functions, assume that

$$z^{\rm tm}(\xi)\Big|_{\xi_n^{\rm tm}} = \frac{d}{d\xi} z^{\rm tm}(\xi)\Big|_{\xi_n^{\rm tm}} = 0 \tag{84}$$

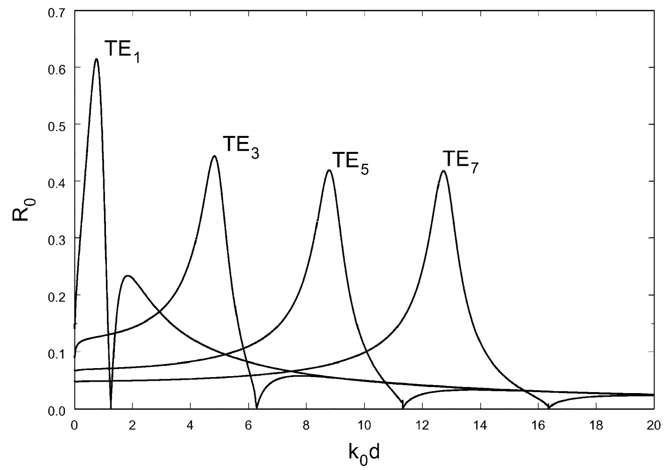

Fig. 2. Coupling amplitude R_0 versus normalized frequency $k_0 d$ for TE modes $n = 1, 3, 5, 7$. The waveguide geometry is shown in Fig. 1, with $\varepsilon_3 = \varepsilon_0$, $\varepsilon_2 = 2.56\varepsilon_0$, $\mu_2 = \mu_3 = \mu_0$.

such that $\xi_n^{\rm tm}$ is a second-order root of the TM transcendental equation. From (17), in region 3 we have

$$\underline{\mathbf{u}}_n^{xx}(\mathbf{r}, \lambda_n) = \underline{\mathbf{u}}_n(x, \xi_n) \frac{e^{im\phi}}{\sqrt{2\pi}} J_m(k_\rho \rho)$$
$$= \hat{\mathbf{x}}\hat{\mathbf{x}} A_5\, e^{-\sqrt{-\xi_n - k_3^2}\, x} J_m(k_\rho \rho) \frac{e^{im\phi}}{\sqrt{2\pi}} \tag{85}$$

where $\lambda_n = \xi_n + k_\rho^2$, $0 \leq k_\rho < \infty$, and $m = \pm 1, \pm 2, \pm 3, \ldots$. Then, (51) leads to

$$\underline{\mathbf{u}}_{n,1}^{xx}(\mathbf{r}, \lambda_n) = \left(\frac{\partial}{\partial \lambda} - c\right) \underline{\mathbf{u}}_n^{xx}(\mathbf{r}, \lambda)\Big|_{\lambda = \lambda_n}$$
$$= A_5\, \hat{\mathbf{x}}\hat{\mathbf{x}} \left[\left(\frac{\rho}{2k_\rho} J_m'(k_\rho \rho) - c J_m(k_\rho \rho)\right) + J_m(k_\rho \rho) \frac{x}{2\sqrt{-\xi_n - k_3^2}}\right] e^{-\sqrt{-\xi_n - k_3^2}\, x} \frac{e^{im\phi}}{2\sqrt{2\pi}}. \tag{86}$$

In rectangular coordinates, from (12) in region 3 we have

$$\underline{\mathbf{u}}_n^{xx}(\mathbf{r}, \lambda_n) = \underline{\mathbf{u}}_n(x, \xi_n) \frac{e^{ik_y y}}{\sqrt{2\pi}} \frac{e^{ik_z z}}{\sqrt{2\pi}}$$
$$= A_5\, \hat{\mathbf{x}}\hat{\mathbf{x}}\, e^{-\sqrt{-\xi_n - k_3^2}\, x} \frac{e^{ik_y y}}{\sqrt{2\pi}} \frac{e^{ik_z z}}{\sqrt{2\pi}} \tag{87}$$

where $\lambda_n = \xi_n + k_y^2 + k_z^2$ and $-\infty < k_{y,z} < \infty$. Then (51) leads to

$$\underline{\mathbf{u}}_{n,1}^{xx}(\mathbf{r}, \lambda_n) = \left(\frac{\partial}{\partial \lambda} - c\right) \underline{\mathbf{u}}_n^{xx}(\mathbf{r}, \lambda)\Big|_{\lambda = \lambda_n}$$
$$= A_5 \hat{\mathbf{x}}\hat{\mathbf{x}} \frac{1}{3} e^{-\sqrt{-\xi_n - k_3^2} x} \frac{e^{ik_y y}}{\sqrt{2\pi}} \frac{e^{ik_z z}}{\sqrt{2\pi}} \times \left(\frac{x}{2\sqrt{-\xi_n - k_3^2}} + \frac{i\, y}{2k_y} + \frac{i\, z}{2k_z} - c\right). \tag{88}$$

Natural modes are obtained from setting $\lambda_n = 0$, leading to $\xi_n + k_y^2 + k_z^2 = 0$. For lossy permittivity ε_2, a numerical example is shown in [16] for the case of low-order TM eigenvalues having multiplicity greater than one [i.e., satisfying (84)], although in that example the value of loss is very high. High-order modes will become degenerate at significantly lower values of loss.

The vertical coordinate improper eigenfunctions are obtained in a similar manner to the proper eigenfunctions by solving (11) subject to (16). Defining

$$m_1^{(3)}(x) \equiv \beta_2 \sin(\beta_2 d) \sin(\beta_3 x) - N^2 \beta_3 \cos(\beta_2 d) \cos(\beta_3 x) \tag{89}$$

$$m_2^{(3)}(x) \equiv \beta_2 \cos(\beta_2 d) \sin(\beta_3 x) + \beta_3 \sin(\beta_2 d) \cos(\beta_3 x) \tag{90}$$

$$m_1^{(2)}(x) \equiv \sin(\beta_2 d) \sin(\beta_2 x) - \cos(\beta_2 d) \cos(\beta_2 x) = -\cos(\beta_2 (d+x)) \tag{91}$$

$$m_2^{(2)}(x) \equiv \sin(\beta_2 d) \cos(\beta_2 x) + \cos(\beta_2 d) \sin(\beta_2 x) = \sin(\beta_2 (d+x)) \tag{92}$$

we obtain

$$u_{\upsilon,\alpha\alpha}^{(3)}(x) = A_7 \, N^2 \, m_2^{(3)}(x) \tag{93}$$

$$u_{\upsilon,x\alpha}^{(3)}(x) = A_8 \, m_1^{(3)}(x) - A_7 \, (N^2 - 1)(ik_\alpha) \sin(\beta_2 d) \sin(\beta_3 x) \tag{94}$$

$$u_{\upsilon,\alpha\alpha}^{(2)}(x) = A_7 \beta_3 \, m_2^{(2)}(x) \tag{95}$$

$$u_{\upsilon,x\alpha}^{(2)}(x) = A_8 \, \beta_3 \, m_1^{(2)}(x) \tag{96}$$

$$u_{\upsilon,xx}^{(3)}(x) = A_9 \, m_1^{(3)}(x) \tag{97}$$

$$u_{\upsilon,xx}^{(2)}(x) = A_9 \, \beta_3 \, m_1^{(2)}(x) \tag{98}$$

and

$$\overline{\nu}_{\upsilon,\alpha\alpha}^{(3)}(x) = A_{10} m_2^{(3)}(x) - A_{11}(N^2 - 1)(ik_\alpha) \cos(\beta_2 d) \sin(\beta_3 x) \tag{99}$$

$$\overline{\nu}_{\upsilon,x\alpha}^{(3)}(x) = A_{11}(-1) m_1^{(3)}(x) \tag{100}$$

$$\overline{\nu}_{\upsilon,\alpha\alpha}^{(2)}(x) = A_{10}(N^2 \beta_3) m_2^{(2)}(x) \tag{101}$$

$$\overline{\nu}_{\upsilon,x\alpha}^{(2)}(x) = A_{11}(-N^2 \beta_3) m_1^{(2)}(x) \tag{102}$$

$$\overline{\nu}_{\upsilon,xx}^{(3)}(x) = A_{12} m_1^{(3)}(x) \tag{103}$$

$$\overline{\nu}_{\upsilon,xx}^{(2)}(x) = A_{12}(N^2 \beta_3) m_1^{(2)}(x) \tag{104}$$

where $\beta_j = -i\gamma_j = -i\sqrt{-\xi_n - k_j^2}$, $j = 2, 3$, and $\xi_\upsilon \in [-k_3^2, \infty)$ is a continuous parameter.

IV. Conclusion

Eigenfunctions of the 3-D Hertzian potential dyadic Green's function operator have been considered. Since the Green's dyadic represents the response due to a general 3-D source, the dyadic eigenfunctions of the Green's operator represent the coupled, 3-D modal fields of the structure. The general theory of dyadic eigenfunctions and dyadic associated functions has been presented, with an example of modal propagation in a grounded dielectric slab environment.

References

[1] D. Marcuse, *Theory of Dielectric Optical Waveguides*. New York: Academic, 1974.

[2] L. B. Felson and N. Marcuvitz, *Radiation and Scattering of Waves*. New York: IEEE Press, 1994.

[3] W. C. Chew, *Waves and Fields in Inhomogeneous Media*. New York: IEEE Press, 1990.

[4] R. E. Collin, *Field Theory of Guided Waves*, 2nd ed. New York: IEEE Press, 1991.

[5] J. R. Wait, *Electromagnetic Waves in Stratified Media*. New York: IEEE Press, 1996.

[6] A. Sommerfeld, *Partial Differential Equations in Physics*. New York: Academic, 1949.

[7] D. G. Dudley, *Mathematical Foundations for Electromagnetic Theory*. New York: IEEE Press, 1994.

[8] G. W. Hanson, A. I. Nosich, and E. M. Kartchevski, "Green's function expansions in dyadic root functions for shielded layered waveguides," *Progr. Electromagn. Res.*, vol. PIER 39, pp. 61–91, 2003.

[9] J. S. Bagby and D. P. Nyquist, "Dyadic Green's functions for integrated electronic and optical circuits," *IEEE Trans. Microwave Theory Tech .*, vol. MTT-35, pp. 206–210, Feb. 1987.

[10] I. V. Lindell, *Methods for Electromagnetic Field Analysis*. Piscataway, NJ: IEEE Press, 1995.

[11] B. Friedman, *Principles and Techniques of Applied Mathematics*. New York: Dover, 1956.

[12] M. A. Naimark, *Linear Differential Operators*. New York: Frederick Ungar, pt. I.

[13] H. Baumgärtel, *Analytic Perturbation Theory for Matrices and Operators*. Boston, MA: Birkhäuser Verlag, 1984.

[14] M. V. Keldysh, "On eigenvalues and eigenfunctions for some classes of nonselfadjoint equations" (in Russian), *Dokl. Akad. Nauk SSSR*, vol. 77, pp. 11–14, 1951.

[15] A. S. Ilyinsky, G. Y. Slepyan, and A. Y. Slepyan, *Propagation, Scattering and Dissipation of Electromagnetic Waves*. London, U.K.: Peregrinus, 1993.

[16] G. W. Hanson and A. B. Yakovlev, "Investigation of mode interaction on planar dielectric waveguides with loss and gain," *Radio Sci.*, vol. 34, no. 6, pp. 1349–1359, Nov.–Dec. 1999.

George W. Hanson (S'85–M'91) was born in Glen Ridge, NJ, in 1963. He received the B.S.E.E. degree from Lehigh University, Bethlehem, PA, the M.S.E.E. degree from Southern Methodist University, Dallas, TX, and the Ph.D. degree from Michigan State University, East Lansing, in 1986, 1988, and 1991, respectively.

From 1986 to 1988, he was a Development Engineer with General Dynamics, Fort Worth, TX, where he worked on radar simulators. He is currently Associate Professor of electrical engineering and computer science at the University of Wisconsin–Milwaukee. His research interests include electromagnetic wave phenomena in layered media, integrated waveguides and transmission lines, mathematical methods in electromagnetics, antennas for wireless communications, and microwave characterization of materials.

Prof. Hanson is a Member of the International Scientific Radio Union (URSI) Commission B, Sigma Xi, the Electromagnetics Academy, and Eta Kappa Nu.