Dyadic Green's Function for a Multilayered Planar Medium—A Dyadic Eigenfunction Approach

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***Abstract*—A new formulation of the dyadic Green's function for a planarly layered medium is presented, based on dyadic eigenfunctions of the Green's function operator. The general development of the dyadic Green's function is shown, resulting in a three-dimensional purely spectral representation. The spectral form is converted to a Hankel-function form using standard techniques, analogous to the sum-of-residues plus branch-cut representation often obtained from the Sommerfeld Green's function. Advantages and disadvantages of both the eigenfunction and Hankel function forms are outlined, and compared to other Green's function representations. Examples of the Green's dyadic for free space and for a grounded dielectric slab environment are provided, and the role of the continuous and discrete spectrum is discussed.**

***Index Terms*—Dyadic, dyadic Green's function, eigenfunction, layered media, natural modes.**

I. Introduction

DYADIC Green's functions are widely used in electromagnetic analysis, and have been the subject of much research over the past several decades. Although space considerations preclude a complete discussion of previous developments here, some of the predominant techniques for developing dyadic Green's functions are exhibited in the following references. In [1]–[3] and related work, dyadic Green's functions are obtained using the (Hansen) vector wavefunctions [4], which are vector eigenfunctions of the curl-curl operator. Related methods utilizing vector modes/eigenfunctions of the structure, not necessarily the Hansen vector wavefunctions, are presented in [2], [5], and [6], and have often been applied to problems involving closed waveguides. Fourier transform domain methods have also been developed, and are particularly useful in layered media. For planarly layered media a double Fourier transform is applied to the infinite coordinates, and the resulting one-dimensional (1-D) vertical-coordinate problem is solved using a variety of methods [7]–[9], and [10]. Three-dimensional Fourier transform methods have also been utilized [11], [12], although the computation can always be reduced to a 2-D Fourier inversion. The traditional methods for determining scalar Green's functions (by eigenfunction expansion/spectral methods, direct differential equation methods, etc.) have been presented in a great number of discipline-specific books and papers (acoustics, thermodynamics, electromagnetics, etc.) as well as in a plethora of applied mathematics books (e.g., [13], [14]). For electromagnetics applications, the material in [2], [15], and [16] provide a good overview of a variety of methods.

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In this paper, a new eigenfunction method of developing dyadic Green's functions is presented, conceptually related to the usual scalar eigenfunction expansion technique from Sturm–Liouville theory. However, in this work recently-introduced dyadic eigenfunctions are utilized, which somewhat simplifies the analysis for the vector problem.

Dyadic eigenfunctions were originally introduced in [17], and later their properties were more fully detailed in [18]. In [17] a source-driven vertically closed structure was considered, and dyadic eigenfunctions were used to interpret the residue contributions of the Sommerfeld-form dyadic Green's function in the event of modal degeneracies. Dyadic eigenfunctions of the vertical-component Green's function operator (i.e., the operator obtained subsequent to Fourier transformation on longitudinal coordinates) were introduced to interpret the residue contributions as dyadic eigenfunctions and dyadic associated functions. In [18] the source-free problem was considered, and dyadic eigenfunctions and associated functions were used to demonstrate modal wave phenomena in an open environment. In this work it is shown how dyadic eigenfunctions can be used to construct dyadic Green's functions in a simple fashion, leading to a formulation that resembles the usual scalar eigenfunction expansion theory.

II. Problem Formulation

Dyadic eigenfunctions, which are eigenfunctions of the dyadic Green's function operator, are examined in detail in [18]. For completeness, the relevant material from [18] is briefly summarized below, followed by the derivation of the dyadic Green's function.

A. Dyadic Eigenfunctions

Assume a laterally-infinite, planarly layered medium inhomogeneous in the x-coordinate, as depicted in Fig. 1. The Hertzian potential Green's dyadic $\underline{\mathbf{g}}(\mathbf{r}, \mathbf{r}')$ provides the Hertzian potential at $\mathbf{r}$ due to an elemental current source at $\mathbf{r}'$, and is the solution of

$$-(\nabla^2 + k^2(x))\underline{\mathbf{g}}(\mathbf{r}, \mathbf{r}') = \underline{\mathbf{I}}\boldsymbol{\delta}(\mathbf{r} - \mathbf{r}') \tag{1}$$

subject to certain boundary and continuity conditions (see [18]), where $k(x) = \omega\sqrt{\mu_0 \varepsilon(x)}$.

Let $\Omega \subseteq \mathbf{R}^3$ be the physical space under consideration, as depicted in Fig. 1. Defining an inner-product as

$$\begin{aligned}\langle \underline{\mathbf{u}}, \underline{\mathbf{v}} \rangle &= \int_\Omega \underline{\mathbf{u}}(\mathbf{r}) : \overline{\underline{\mathbf{v}}}(\mathbf{r}) d\Omega \\ &= \int_\Omega \sum_{i,j=1}^{3} u_{ij}(\mathbf{r}) \overline{v}_{ij}(\mathbf{r}) d\Omega\end{aligned} \tag{2}$$

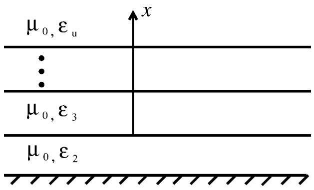

Fig. 1. Multilayered planar environment.

utilizing the double-dot product notation [19], where the overbar indicates complex conjugation, the space $H = \mathbf{L}^2(\Omega)$ is the space of functions such that

$$\|\underline{\mathbf{u}}\|^2 = \langle \underline{\mathbf{u}}, \underline{\mathbf{u}} \rangle < \infty. \tag{3}$$

Associated with the dyadic Green's function problem is the dyadic eigenfunction problem in H (in a generalized sense, including improper eigenfunctions [18])

$$-(\nabla^2 + k^2(x))\underline{\mathbf{u}}_n(\mathbf{r}) = \lambda_n \underline{\mathbf{u}}_n(\mathbf{r}) \tag{4}$$

which has the separation of variables solution

$$\underline{\mathbf{u}}_n(\mathbf{r}) = \underline{\mathbf{u}}_n(x, \xi_n) \frac{e^{jk_y y}}{\sqrt{2\pi}} \frac{e^{jk_z z}}{\sqrt{2\pi}} \tag{5}$$

where k_y, $k_z \in (-\infty, \infty)$ are continuous variables and $\lambda_n = \xi_n + k_y^2 + k_z^2$. The x-coordinate dyadic eigenfunctions $\underline{\mathbf{u}}_n(x, \xi_n) = \underline{\mathbf{u}}_n^{\beta\beta}(x, \xi_n)$, $\beta\beta = xx, yy, zz$, satisfy

$$-\left(\frac{d^2}{dx^2} + k^2(x)\right) \underline{\mathbf{u}}_n^{\beta\beta}(x, \xi_n) = \xi_n \underline{\mathbf{u}}_n^{\beta\beta}(x). \tag{6}$$

The dyadic eigenfunctions $\underline{\mathbf{u}}_n^{\beta\beta}(x, \xi_n)$ may be proper (vanish at vertical infinity), in which case n and ξ_n are discrete variables, leading to TE and TM transcendental equations $z^{te}(\xi_n) = 0$ and $z^{tm}(\xi_n) = 0$, respectively, or improper (related to radiation modes), where n and ξ_n are continuous variables, in which case the eigenfunctions are merely bounded at vertical infinity. The dyadic eigenfunctions have the form [18]

$$\begin{aligned}
&\underline{\mathbf{u}}_{n_e}^{xx}(x) = \widehat{\mathbf{x}}\widehat{\mathbf{x}} u_{n_e,xx}, \quad \underline{\mathbf{u}}_{n_h}^{xx}(x) = \underline{\mathbf{0}} \\
&\underline{\mathbf{u}}_{n_e}^{yy}(x) = \widehat{\mathbf{x}}\widehat{\mathbf{y}} u_{n_e,xy}, \quad \underline{\mathbf{u}}_{n_h}^{yy}(x) = \widehat{\mathbf{x}}\widehat{\mathbf{y}} u_{n_h,xy} + \widehat{\mathbf{y}}\widehat{\mathbf{y}} u_{n_h,yy} \\
&\underline{\mathbf{u}}_{n_e}^{zz}(x) = \widehat{\mathbf{x}}\widehat{\mathbf{z}} u_{n_e,xz} \\
&\underline{\mathbf{u}}_{n_h}^{zz}(x) = \widehat{\mathbf{x}}\widehat{\mathbf{z}} u_{n_h,xz} + \widehat{\mathbf{z}}\widehat{\mathbf{z}} u_{n_h,zz}
\end{aligned} \tag{7}$$

where n_e and n_h indicate $E(\mathrm{TM})$ and $H(\mathrm{TE})$ modes, respectively.

The adjoint eigenvalue problem is

$$-(\nabla^2 + \overline{k}^2(x))\underline{\mathbf{v}}_n(\mathbf{r}) = \lambda_n^* \underline{\mathbf{v}}_n(\mathbf{r}) \tag{8}$$

where $\lambda_n^* = \overline{\lambda_n}$, leading to adjoint eigenfunctions

$$\underline{\mathbf{v}}_n^{\beta\beta}(\mathbf{r}) = \underline{\mathbf{v}}_n^{\beta\beta}(x, \xi_n) \frac{e^{jk_y y}}{\sqrt{2\pi}} \frac{e^{jk_z z}}{\sqrt{2\pi}} \tag{9}$$

with [18]

$$\begin{aligned}
&\overline{\underline{\mathbf{v}}}_{n_e}^{xx}(x) = \widehat{\mathbf{x}}\widehat{\mathbf{x}} \overline{v}_{n_e,xx}, \quad \overline{\underline{\mathbf{v}}}_{n_h}^{xx}(x) = \underline{\mathbf{0}} \\
&\overline{\underline{\mathbf{v}}}_{n_e}^{yy}(x) = \widehat{\mathbf{x}}\widehat{\mathbf{y}} \overline{v}_{n_e,xy} + \widehat{\mathbf{y}}\widehat{\mathbf{y}} \overline{v}_{n_e,yy}, \quad \overline{\underline{\mathbf{v}}}_{n_h}^{yy}(x) = \widehat{\mathbf{y}}\widehat{\mathbf{y}} \overline{v}_{n_h,yy} \\
&\overline{\underline{\mathbf{v}}}_{n_e}^{zz}(x) = \widehat{\mathbf{x}}\widehat{\mathbf{z}} \overline{v}_{n_e,xz} + \widehat{\mathbf{z}}\widehat{\mathbf{z}} \overline{v}_{n_e,zz} \\
&\overline{\underline{\mathbf{v}}}_{n_h}^{zz}(x) = \widehat{\mathbf{z}}\widehat{\mathbf{z}} \overline{v}_{n_h,zz}.
\end{aligned} \tag{10}$$

The orthogonality relationships are presented in [18]

$$\begin{aligned}
(\lambda_n - \lambda_m) \left\langle \underline{\mathbf{u}}_n^{\beta\beta}, \underline{\mathbf{v}}_m^{\beta\beta} \right\rangle &= 0, \quad \beta = x, y, z \\
\left\langle \underline{\mathbf{u}}_n^{\alpha\alpha}, \underline{\mathbf{v}}_m^{\beta\beta} \right\rangle &= 0, \quad \alpha, \beta = x, y, z \\
&\alpha \neq \beta.
\end{aligned} \tag{11}$$

Eigenfunctions proper in the vertical coordinate are normalized as

$$\begin{aligned}
\left\langle \underline{\mathbf{u}}_n^{\beta\beta}, \underline{\mathbf{v}}_m^{\beta\beta} \right\rangle &= \delta(k_y - k_y')\delta(k_z - k_z')\delta_{nm} \\
\beta &= x, y, z
\end{aligned} \tag{12}$$

where $\delta(\nu - \nu')$ is the Dirac delta function and δ_{nm} is the Kronecker delta function. Eigenfunctions improper in the vertical coordinate are normalized as

$$\begin{aligned}
\left\langle \underline{\mathbf{u}}_n^{\beta\beta}, \underline{\mathbf{v}}_m^{\beta\beta} \right\rangle &= \delta(k_y - k_y')\delta(k_z - k_z')\delta(\xi - \xi') \\
\beta &= x, y, z.
\end{aligned} \tag{13}$$

B. Dyadic Green's Function

To derive the dyadic eigenfunction expansion of the Hertzian potential dyadic Green's function, assume that the dyadic eigenfunctions form an orthonormal basis of H. Although this fact has not been proved rigorously, a theorem in [20] regarding nonselfadjoint Strum–Liouville problems can be applied to the xx, yy, and zz-component scalar problems, proving completeness of the eigenfunctions when eigenvalues have unit multiplicity. For nontrivial modal degeneracies (associated with multiplicity $m > 1$ eigenvalues lacking m independent eigenvectors) the root system must be utilized to form a complete set (i.e., a basis) [17]. In this case the development of the dyadic Green's function is considerably more complicated, and will be considered elsewhere.

Assume a layered medium as depicted in Fig. 1, where the uppermost layer has wavenumber k_u. The development of the spectral form of the Green's function follows the usual procedure, elevated to dyadic level. Since the operator of the dyadic Green's function has both purely continuous and mixed discrete/continuous spectra, the spectral expansion of the Hertzian potential dyadic Green's function is

$$\begin{aligned}
\underline{\mathbf{g}}(\mathbf{r}, \mathbf{r}') = &\sum_{\beta\beta = xx,yy,zz} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_y dk_z \\
&\times \left\{ \sum_{n_e=1}^{N_e} a_{n_e}^{\beta\beta} \underline{\mathbf{u}}_{n_e}^{\beta\beta}(\mathbf{r}) + \sum_{n_h=1}^{N_h} a_{n_h}^{\beta\beta} \underline{\mathbf{u}}_{n_h}^{\beta\beta}(\mathbf{r}) \right. \\
&\quad + \int_{-k_u^2}^{\infty} a_{\xi_e}^{\beta\beta}(\xi) \underline{\mathbf{u}}_{\xi_e}^{\beta\beta}(\mathbf{r}) d\xi \\
&\quad \left. + \int_{-k_u^2}^{\infty} a_{\xi_h}^{\beta\beta}(\xi) \underline{\mathbf{u}}_{\xi_h}^{\beta\beta}(\mathbf{r}) d\xi \right\}
\end{aligned} \tag{14}$$

where $\xi_e(\xi_h)$ indicates $E(H)$ modes, and $N_{e,h} = N_{e,h}(\omega)$ is the number of above cutoff surface wave modes. It is clear from [18] that the continuous spectrum associated with an unbounded vertical coordinate lies along $\xi \in \left[-k_u^2, \infty\right)$, since the continuous eigenfunctions contain the factor $\sqrt{\xi + k_i^2}$, where i represents the ith material layer [see, e.g., (40)–(45)]. Branch points associated with interior (vertically-finite) layers are removable, as discussed in [3]. If the lowermost layer is a semi-infinite dielectric with wavenumber k_l instead of a perfect conductor, the continuous spectrum would also involve an integration along $\left[-k_l^2, \infty\right)$, associated with radiation into the lowermost semi-infinite medium. In what follows, we assume the presence of a perfect (lower) ground plane for simplicity.

Utilizing (1) and (4) leads to

$$\begin{aligned}
&-(\nabla^2 + k^2(x))\underline{\mathbf{g}}(\mathbf{r},\mathbf{r}') \\
&= \sum_{\beta\beta=xx,yy,zz} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_y dk_z \\
&\quad \times \left\{ \sum_{n_e=1}^{N_e} a_{n_e}^{\beta\beta}\lambda_{n_e}\underline{\mathbf{u}}_{n_e}^{\beta\beta}(\mathbf{r}) + \sum_{n_h=1}^{N_h} a_{n_h}^{\beta\beta}\lambda_{n_h}\underline{\mathbf{u}}_{n_h}^{\beta\beta}(\mathbf{r}) \right. \\
&\quad + \int_{-k_u^2}^{\infty} a_{\xi_e}^{\beta\beta}(\xi)\lambda_{\xi_e}\underline{\mathbf{u}}_{\xi_e}^{\beta\beta}(\mathbf{r})d\xi \\
&\quad \left. + \int_{-k_u^2}^{\infty} a_{\xi_h}^{\beta\beta}(\xi)\lambda_{\xi_h}\underline{\mathbf{u}}_{\xi_h}^{\beta\beta}(\mathbf{r})d\xi \right\} = \underline{\mathbf{I}}\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}'). \qquad (15)
\end{aligned}$$

Taking the inner-product with the adjoint eigenfunctions

$$\begin{aligned}
&\sum_{\beta\beta=xx,yy,zz} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dk_y dk_z \\
&\quad \times \left\{ \sum_{n_e=1}^{N_e} a_{n_e}^{\beta\beta}\lambda_{n_e}\left\langle \underline{\mathbf{u}}_{n_e}^{\beta\beta}, \underline{\mathbf{v}}_k^{\alpha\alpha}\right\rangle \right. \\
&\quad + \sum_{n_h=1}^{N_h} a_{n_h}^{\beta\beta}\lambda_{n_h}\left\langle \underline{\mathbf{u}}_{n_h}^{\beta\beta}, \underline{\mathbf{v}}_k^{\alpha\alpha}\right\rangle \\
&\quad + \int_{-k_u^2}^{\infty} a_{\xi_e}^{\beta\beta}(\xi)\lambda_{\xi_e}\left\langle \underline{\mathbf{u}}_{\xi_e}^{\beta\beta}, \underline{\mathbf{v}}_k^{\alpha\alpha}\right\rangle d\xi \\
&\quad \left. + \int_{-k_u^2}^{\infty} a_{\xi_h}^{\beta\beta}(\xi)\lambda_{\xi_h}\left\langle \underline{\mathbf{u}}_{\xi_h}^{\beta\beta}, \underline{\mathbf{v}}_k^{\alpha\alpha}\right\rangle d\xi \right\} \\
&= \left\langle \underline{\mathbf{I}}\boldsymbol{\delta}(\mathbf{r}-\mathbf{r}'), \underline{\mathbf{v}}_k^{\alpha\alpha}\right\rangle = \overline{v}_{k,\alpha\alpha}(\mathbf{r}') \qquad (16)
\end{aligned}$$

and exploiting orthonormality leads to

$$\begin{aligned}
\underline{\mathbf{g}}(\mathbf{r},\mathbf{r}') &= \sum_{\beta\beta=xx,yy,zz} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \\
&\quad \times \left\{ \sum_{n_e=1}^{N_e} \frac{\underline{\mathbf{u}}_{n_e}^{\beta\beta}(x)\overline{v}_{n_e,\beta\beta}(x')}{k_\rho^2 + \xi_{n_e}} \right. \\
&\quad + \sum_{n_h=1}^{N_h} \frac{\underline{\mathbf{u}}_{n_h}^{\beta\beta}(x)\overline{v}_{n_h,\beta\beta}(x')}{k_\rho^2 + \xi_{n_h}} \\
&\quad + \int_{-k_u^2}^{\infty} \frac{\underline{\mathbf{u}}_{\xi_e}^{\beta\beta}(x)\overline{v}_{\xi_e,\beta\beta}(x')}{k_\rho^2 + \xi_e} d\xi_e \\
&\quad \left. + \int_{-k_u^2}^{\infty} \frac{\underline{\mathbf{u}}_{\xi_h}^{\beta\beta}(x)\overline{v}_{\xi_h,\beta\beta}(x')}{k_\rho^2 + \xi_h} d\xi_h \right\} \\
&\quad \times \frac{e^{jk_y(y-y')}}{2\pi}\frac{e^{jk_z(z-z')}}{2\pi} dk_y dk_z \qquad (17)
\end{aligned}$$

which is a pure spectral expansion, involving both continuous spectra and mixed discrete/continuous spectra. This representation is fundamental, in the sense that the Green's function is expressed directly in terms of the 3-D spectral elements of the operator associated with the physical structure. It is clear that there are no true discrete "modes" of the structure, which makes physical sense since the structure is laterally unbounded. Columns of the dyadic eigenfunctions are vector eigenmodes, and columns of the constructed Green's dyadic are vector potentials.

It can be shown that the off-diagonal elements of $\underline{\mathbf{u}}_n^{\alpha\alpha}(x)$ or $\underline{\mathbf{v}}_n^{\alpha\alpha}(x)$, $\alpha = y, z$, contain a multiplicative factor jk_α arising from the boundary conditions that couple potential components at a dielectric interface (see e.g., [18, eq. (61)]). In what follows it is useful to express this multiplicative factor explicitly in (17) (i.e., move this multiplicative factor from $\underline{\mathbf{u}}_n^{\alpha\alpha}(x)$ or $\underline{\mathbf{v}}_n^{\alpha\alpha}(x)$ to the off-diagonal Green's dyadic entries), resulting in, with a slight abuse of notation

$$\begin{aligned}
\underline{\mathbf{g}}(\mathbf{r},\mathbf{r}') &= \sum_{\beta\beta=xx,yy,zz} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \widetilde{\underline{\mathbf{D}}}^{\beta\beta} \\
&\quad \cdot \left\{ \sum_{n_e=1}^{N_e} \frac{\underline{\mathbf{u}}_{n_e}^{\beta\beta}(x)\overline{v}_{n_e,\beta\beta}(x')}{k_\rho^2 + \xi_{n_e}} \right. \\
&\quad + \sum_{n_h=1}^{N_h} \frac{\underline{\mathbf{u}}_{n_h}^{\beta\beta}(x)\overline{v}_{n_h,\beta\beta}(x')}{k_\rho^2 + \xi_{n_h}} \\
&\quad + \int_{-k_u^2}^{\infty} \frac{\underline{\mathbf{u}}_{\xi_e}^{\beta\beta}(x)\overline{v}_{\xi_e,\beta\beta}(x')}{k_\rho^2 + \xi_e} d\xi_e \\
&\quad \left. + \int_{-k_u^2}^{\infty} \frac{\underline{\mathbf{u}}_{\xi_h}^{\beta\beta}(x)\overline{v}_{\xi_h,\beta\beta}(x')}{k_\rho^2 + \xi_h} d\xi_h \right\} \\
&\quad \times \frac{e^{jk_y(y-y')}}{2\pi}\frac{e^{jk_z(z-z')}}{2\pi} dk_y dk_z \qquad (18)
\end{aligned}$$

where

$$\widetilde{\underline{\mathbf{D}}}^{xx} = \hat{\mathbf{x}}\hat{\mathbf{x}}, \quad \widetilde{\underline{\mathbf{D}}}^{\alpha\alpha} = \hat{\mathbf{x}}\hat{\mathbf{x}}(jk_\alpha) + \hat{\boldsymbol{\alpha}}\hat{\boldsymbol{\alpha}} \qquad (19)$$

$\alpha = y, z$, which merely applies the multiplicative factor (jk_α), $\alpha = y, z$, to the off-diagonal entries of the Green's dyadic. Using the correspondence $jk_\alpha \leftrightarrow \partial/(\partial\alpha)$, we can move $\widetilde{\underline{\mathbf{D}}}^{\beta\beta}$ outside of the double integral, resulting in

$$\begin{aligned}
\underline{\mathbf{g}}(\mathbf{r},\mathbf{r}') &= \sum_{\beta\beta=xx,yy,zz} \underline{\mathbf{D}}^{\beta\beta} \\
&\quad \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \sum_{n_e=1}^{N_e} \frac{\underline{\mathbf{u}}_{n_e}^{\beta\beta}(x)\overline{v}_{n_e,\beta\beta}(x')}{k_\rho^2 + \xi_{n_e}} \right. \\
&\quad + \sum_{n_h=1}^{N_h} \frac{\underline{\mathbf{u}}_{n_h}^{\beta\beta}(x)\overline{v}_{n_h,\beta\beta}(x')}{k_\rho^2 + \xi_{n_h}} \\
&\quad + \int_{-k_u^2}^{\infty} \frac{\underline{\mathbf{u}}_{\xi_e}^{\beta\beta}(x)\overline{v}_{\xi_e,\beta\beta}(x')}{k_\rho^2 + \xi_e} d\xi_e \\
&\quad \left. + \int_{-k_u^2}^{\infty} \frac{\underline{\mathbf{u}}_{\xi_h}^{\beta\beta}(x)\overline{v}_{\xi_h,\beta\beta}(x')}{k_\rho^2 + \xi_h} d\xi_h \right\} \\
&\quad \times \frac{e^{jk_y(y-y')}}{2\pi}\frac{e^{jk_z(z-z')}}{2\pi} dk_y dk_z \qquad (20)
\end{aligned}$$

where

$$\underline{\mathbf{D}}^{xx} = \widehat{\mathbf{x}}\widehat{\mathbf{x}}, \quad \underline{\mathbf{D}}^{\alpha\alpha} = \widehat{\mathbf{x}}\widehat{\mathbf{x}}\frac{\partial}{\partial\alpha} + \widehat{\boldsymbol{\alpha}}\widehat{\boldsymbol{\alpha}}. \tag{21}$$

Alternative to (20), using standard manipulations the double integral over (k_y, k_z) can be eliminated from (17). For instance, consider a generic (partially) discrete term (the normalization constants associated with the eigenfunctions can be made independent of k_y and k_z)

$$\begin{aligned}\underline{\boldsymbol{\chi}}^d &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_n \frac{\underline{\mathbf{u}}_n^{\beta\beta}(x)\overline{v}_{n,\beta\beta}(x')}{k_y^2+k_z^2+\xi_n} \\ &\quad \times \frac{e^{jk_y(y-y')}}{2\pi}\frac{e^{jk_z(z-z')}}{2\pi}dk_y dk_z \\ &= 2\pi j\int_{-\infty}^{\infty}\sum_n \frac{\underline{\mathbf{u}}_n^{\beta\beta}(x)\overline{v}_{n,\beta\beta}(x')}{-2\sqrt{-\xi_n-k_z^2}} \\ &\quad \times \frac{e^{-j\sqrt{-\xi_n-k_z^2}(y-y')}}{2\pi}\frac{e^{jk_z(z-z')}}{2\pi}dk_z \\ &= \frac{1}{4j}\sum_n \underline{\mathbf{u}}_n^{\beta\beta}(x)\overline{v}_{n,\beta\beta}(x')H_0^{(2)}(\sqrt{-\xi_n}\rho)\end{aligned} \tag{22}$$

where $\rho = ((y-y')^2+(z-z')^2)^{1/2}$, $\underline{\mathbf{u}}_n^{\beta\beta}(x) = \underline{\mathbf{u}}_n^{\beta\beta}(x,\xi_n)$, and where ξ_n is the x–coordinate eigenfunction which satisfies $z^{te}(\xi_n) = 0$ or $z^{tm}(\xi_n) = 0$. Note that $-k_L \leq \xi_n \leq -k_S$ for lossless media, where k_L and k_S are the largest and smallest wavenumbers. For lossy media k_L and k_S are complex-valued, and ξ_n resides in the second quadrant of the complex plane.

Similarly, for a generic purely continuous spectrum term we obtain

$$\begin{aligned}\underline{\boldsymbol{\chi}}^c &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-k_u^2}^{\infty} \frac{\underline{\mathbf{u}}_{\xi_e}^{\beta\beta}(x)\overline{v}_{\xi,\beta\beta}(x')}{k_\rho^2+\xi} \\ &\quad \times \frac{e^{jk_y(y-y')}}{2\pi}\frac{e^{jk_z(z-z')}}{2\pi}d\xi dk_y dk_z \\ &= \frac{1}{4j}\int_{-k_3^2}^{\infty}\underline{\mathbf{u}}_\xi^{\beta\beta}(x)\overline{v}_{\xi,\beta\beta}(x')H_0^{(2)}(\sqrt{-\xi}\rho)d\xi\end{aligned} \tag{23}$$

where $\underline{\mathbf{u}}_\xi^{\beta\beta}(x) = \underline{\mathbf{u}}_\xi^{\beta\beta}(x,\xi)$ with $\xi \in \left(-k_u^2, \infty\right]$. The change of variables $\varsigma = j\sqrt{\xi}$, $d\varsigma = j/\left(2\sqrt{\xi}\right)d\xi$ leads to the usual hyperbolic branch-cut integration

$$\underline{\boldsymbol{\chi}}^c = \frac{j}{2}\int_{-k_u}^{j\infty}\underline{\mathbf{u}}_\varsigma^{\beta\beta}(x)\overline{v}_{\varsigma,\beta\beta}(x')H_0^{(2)}(\varsigma\rho)\varsigma d\varsigma \tag{24}$$

where $\underline{\mathbf{u}}_\varsigma^{\beta\beta}(x) = \underline{\mathbf{u}}_\varsigma^{\beta\beta}\left(x, -\varsigma^2\right)$ and the integration path consists of $\varsigma \in [-k_u, 0]$ on the negative-real axis and $\varsigma \in [0, \infty)$ along the positive-imaginary axis. Alternatively, the change of variables $\varkappa = \xi + k_u^2$, $d\varkappa = d\xi$ in (23) leads to the representation

$$\underline{\boldsymbol{\chi}}^c = \frac{1}{4j}\int_0^{\infty}\underline{\mathbf{u}}_\varkappa^{\beta\beta}(x)\overline{v}_{\varkappa,\beta\beta}(x')H_0^{(2)}\left(\sqrt{k_u^2-\varkappa}\rho\right)d\varkappa \tag{25}$$

where $\underline{\mathbf{u}}_\varkappa^{\beta\beta}(x) = \underline{\mathbf{u}}_\varkappa^{\beta\beta}\left(x, \varkappa - k_u^2\right)$ with the path of integration being the positive real- $\varkappa$ axis, $\varkappa \in [0, \infty)$.

Therefore, (20) is equivalent to

$$\begin{aligned}\underline{\mathbf{g}}(\mathbf{r},\mathbf{r}') &= \frac{1}{4j}\sum_{\beta\beta=xx,yy,zz}\underline{\mathbf{D}}^{\beta\beta} \\ &\quad \cdot \left\{\sum_{n_e=1}^{N_e}\underline{\mathbf{u}}_{n_e}^{\beta\beta}(x)\overline{v}_{n_e,\beta\beta}(x')H_0^{(2)}\left(\sqrt{-\xi_{n_e}}\rho\right)\right. \\ &\quad + \sum_{n_h=1}^{N_h}\underline{\mathbf{u}}_{n_h}^{\beta\beta}(x)\overline{v}_{n_h,\beta\beta}(x')H_0^{(2)}\left(\sqrt{-\xi_{n_h}}\rho\right) \\ &\quad - 2\int_{-k_u}^{j\infty}\underline{\mathbf{u}}_{\varsigma_e}^{\beta\beta}(x)\overline{v}_{\varsigma_e,\beta\beta}(x')H_0^{(2)}(\varsigma_e\rho)\varsigma_e d\varsigma_e \\ &\quad \left. - 2\int_{-k_u}^{j\infty}\underline{\mathbf{u}}_{\xi_h}^{\beta\beta}(x)\overline{v}_{\varsigma_h,\beta\beta}(x')H_0^{(2)}(\varsigma_h\rho)\varsigma_h d\varsigma_h\right\}\end{aligned} \tag{26}$$

where for the continuous spectrum terms the representation (24) was used, although any of the forms (23)–(25) could be used. From (26), (7), and (10), note that the off-diagonal entries involve both TE and TM modes, whereas the diagonal entries involve one mode class (TM for the xx component andTE for the yy and zz components).

The representation (26) is equivalent to the residue-plus-branch-cut form usually obtained from complex-plane analysis of the Sommerfeld Green's function. The discrete terms in (26), which are equivalent to the residue contributions, are not "modes" of the structure, if a mode is defined as the natural, source-free response. Since the origin is in the domain of the Helmholtz operator, the natural mode in cylindrical coordinates contains a first-kind Bessel function [18]. The Hankel function is merely obtained from the Bessel function upon altering the limits of integration in the spectral response. Therefore, while (26) has a nice physical interpretation in terms of discrete and continuous cylindrically propagating waves, it is not a "sum of modes" representation as is commonly stated.

The form (26) is the simplest form for direct computation. However, if one needs to integrate a current density with the Green's function to obtain the potential [see, e.g., (28)], the presence of the Hankel function will make it difficult to evaluate the spatial integral in closed-form. In this case, it may be better to use the purely spectral forms (18) or (20), although one must evaluate the resulting Sommerfeld integrals.

In general, from a computational point of view, the separation of the Green's function into terms associated with surface waves and terms associated with continuous radiation has both advantages and disadvantages. One advantage is that the integrals [in either (17) or (26)] do not contain pole singularities associated with the background waveguide. A disadvantage of this separation is obviously the need to know the eigenvalues of the structure. Note, however, that the extraction of surface wave poles is often done when using the Sommerfeld form to obtain a smoother integrand, a method which obviously necessitates knowledge of the eigenvalues. Similarly, one needs to know the eigenvalues to use some of the quasiclosed-form (complex image) methods for fast evaluation of the Sommerfeld integral. Therefore, the need to know the eigenvalues is fairly common to problems of layered media.

Furthermore, there are several situations of practical interest when the eigenfunction form or Hankel function form is particularly useful. For example, most often in practical situations the electrical thickness of the dielectric layer is small (e.g., printed circuit antennas, where thick layers lead to undesirable crosstalk, scan blindness, and diffraction effects). In this case, often only one mode, or just a few modes, are above cutoff. In these situations the discrete terms in (26) can be easily computed. Also, for field points vertically far above the dielectric (x sufficiently large) the discrete terms in (26) can be neglected, and near to the dielectric and laterally far from the source, the integral contributions in (26) can be neglected. This is, of course, also true of the corresponding terms in (17). In these cases the computation is simplified from the general case, and, further, asymptotic analysis can be performed leading to closed form results. A discussion of numerical and asymptotic techniques related to Green's functions for layered media is provided in [21] and references therein.

The electric and magnetic fields due to the electric Hertzian potential are [2]

$$\begin{aligned}\mathbf{E}(\mathbf{r}) &= (k^2(x) + \nabla\nabla\cdot)\boldsymbol{\pi}_e(\mathbf{r})\\ \mathbf{H}(\mathbf{r}) &= j\omega\varepsilon(x)\nabla\times\boldsymbol{\pi}_e(\mathbf{r})\end{aligned}\tag{27}$$

where

$$\boldsymbol{\pi}_e(\mathbf{r}) = \int_\Omega \underline{\mathbf{g}}(\mathbf{r},\mathbf{r}')\cdot\frac{\mathbf{J}(\mathbf{r}')}{j\omega\varepsilon(x')}d\Omega'.\tag{28}$$

The dyadic Green's functions for the electric and magnetic fields such that

$$\mathbf{E}(\mathbf{r}) = \int_\Omega \underline{\mathbf{g}}^{e,e}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')d\Omega'\tag{29}$$

$$\mathbf{H}(\mathbf{r}) = \int_\Omega \underline{\mathbf{g}}^{h,e}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')d\Omega'\tag{30}$$

are given by

$$\begin{aligned}\underline{\mathbf{g}}^{e,e}(\mathbf{r},\mathbf{r}') =& P.V.\left(\frac{k^2(x)+\nabla\nabla\cdot}{j\omega\varepsilon(x')}\right)\underline{\mathbf{g}}(\mathbf{r},\mathbf{r}')\\ &- \hat{\mathbf{x}}\hat{\mathbf{x}}\frac{\delta(\mathbf{r}-\mathbf{r}')}{j\omega\varepsilon(x')}\end{aligned}\tag{31}$$

$$\underline{\mathbf{g}}^{h,e}(\mathbf{r},\mathbf{r}') = \frac{\varepsilon(x)}{\varepsilon(x')}\nabla\times\underline{\mathbf{g}}(\mathbf{r},\mathbf{r}')\tag{32}$$

where $P.V.$ indicates that the spatial volume integral to obtain the field from the source must be evaluated in the principle-value sense (due to the second derivative operator) [22]. The discrete summations in (26) are finite, and involve well-behaved functions, and so the continuous spectrum terms clearly contain the source-point singularity, leading to the necessity of the $P.V.$ integration in (31).

The Hertzian potential Green's dyadic for free space is easily found to be

$$\begin{aligned}\underline{\mathbf{g}}(\mathbf{r},\mathbf{r}') =& \underline{\mathbf{I}}\frac{e^{jkR}}{4\pi R}\\ =& \underline{\mathbf{I}}\frac{j}{2}\left\{\int_{-k}^{j\infty} u_{\varsigma_e}(x)\overline{v}_{\varsigma_e}(x')H_0^{(2)}(\varsigma_e\rho)\varsigma_e d\varsigma_e\right.\\ &\left.+\int_{-k}^{j\infty} u_{\varsigma_h}(x)\overline{v}_{\varsigma_h}(x')H_0^{(2)}(\varsigma_h\rho)\varsigma_h d\varsigma_h\right\}\end{aligned}\tag{33}$$

where

$$u_{\varsigma_e}(x) = \frac{\cos\beta x}{2\pi|\beta|} = \overline{v}_{\varsigma_e}(x)\tag{34}$$

$$u_{\varsigma_h}(x) = \frac{\sin\beta x}{2\pi|\beta|} = \overline{v}_{\varsigma_h}(x)\tag{35}$$

and $\beta = \sqrt{k^2-\varsigma^2}$ (one method to derive (33) is by finding the Hertzian potential Green's dyadic for a parallel-plate waveguide, homogeneously-filled, and let the waveguide walls recede to infinity). Alternatively, for the representation (25)

$$\begin{aligned}\frac{e^{jkR}}{4\pi R} =& \frac{1}{4j}\left\{\int_0^\infty u_{\varkappa_e}(x)\overline{v}_{\varkappa_e}(x')H_0^{(2)}\left(\sqrt{k^2-\varkappa_e}\rho\right)d\varkappa_e\right.\\ &\left.+\int_0^\infty u_{\varkappa_h}(x)\overline{v}_{\varkappa_h}(x')H_0^{(2)}\left(\sqrt{k^2-\varkappa_e}\rho\right)d\varkappa_h\right\}\end{aligned}\tag{36}$$

where $u_\varkappa$ and $\overline{v}_\varkappa$ are the same as in (34) and (35) with $\beta = \sqrt{\varkappa}$. Using a simple trigonometric identity leads to the various forms

$$\begin{aligned}\frac{e^{jkR}}{4\pi R} =& \frac{j}{4\pi}\int_{-k}^{j\infty}\frac{\cosh\left(\sqrt{\varsigma^2-k^2}(x-x')\right)}{|\sqrt{\varsigma^2-k^2}|}\\ &\times H_0^{(2)}(\varsigma\rho)\varsigma d\varsigma \qquad (37)\\ =& \frac{1}{8\pi j}\int_0^\infty\frac{\cos\left(\sqrt{\varkappa}(x-x')\right)}{\sqrt{\varkappa}}\\ &\times H_0^{(2)}\left(\sqrt{k^2-\varkappa}\rho\right)d\varkappa \qquad (38)\\ =& \frac{1}{8\pi j}\int_{-\infty}^{\infty}e^{j\varsigma(x-x')}H_0^{(2)}\left(\sqrt{k^2-\varsigma^2}\rho\right)d\varsigma. \qquad (39)\end{aligned}$$

To remove the need for the $P.V.$ integration in (31), elements of the free-space Hertzian-potential Green's dyadic (33) can be subtracted from the continuous spectrum diagonal elements of (26), eliminating the source-point singularity from (26). The spatial form of the free-space potential, $e^{jkR}/(4\pi R)$, can then be "added-back" to these same terms, and the electric Green's dyadic formed by typical methods used in free space [2], [22].

III. EXAMPLE

In [18], the dyadic eigenfunctions for a grounded slab waveguide typical of printed antenna structures (see Fig. 2) were presented; see [18, eqs. (55)–(68) and (73)–(83)] for the mixed continuous/discrete eigenfunctions. The purely continuous eigenfunctions are also given in [18], although here a different (although equivalent) form is utilized in which the TE and TM components are separated, which is necessary for their application in the Green's function.

Define

$$\begin{aligned}m_1^{(3)}(x) \equiv& \beta_2\sin(\beta_2 d)\sin(\beta_3 x)\\ &- N^2\beta_3\cos(\beta_2 d)\cos(\beta_3 x)\end{aligned}\tag{40}$$

$$\begin{aligned}m_2^{(3)}(x) \equiv& \beta_2\cos(\beta_2 d)\sin(\beta_3 x)\\ &+ \beta_3\sin(\beta_2 d)\cos(\beta_3 x)\end{aligned}\tag{41}$$

$$\begin{aligned}m_3^{(3)}(x) \equiv& \beta_2\cos(\beta_2 d)\cos(\beta_3 x)\\ &- \beta_3\sin(\beta_2 d)\sin(\beta_3 x)\end{aligned}\tag{42}$$

$$\begin{aligned}m_4^{(3)}(x) \equiv& N^2\beta_3\cos(\beta_2 d)\sin(\beta_3 x)\\ &+ \beta_2\sin(\beta_2 d)\cos(\beta_3 x)\end{aligned}\tag{43}$$

$$m_1^{(2)}(x) \equiv -\cos(\beta_2(d+x))\tag{44}$$

$$m_2^{(2)}(x) \equiv \sin(\beta_2(d+x))\tag{45}$$

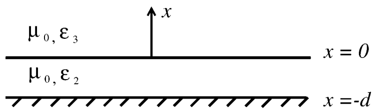

Fig. 2. Grounded dielectric slab.

where for the representation (23), $\beta_j = -j\gamma_j = \sqrt{\xi + k_j^2}$, $j = 2, 3$ and $\xi \in [-k_3^2, \infty)$ is a continuous parameter. For the representation (24), $\beta_i = \sqrt{k_i^2 - \varsigma^2}$ where $\varsigma \in [-k_3, 0]$ and $\varsigma \in [0, j\infty)$, and for (25) we have $\beta_i = \sqrt{\varsigma - k_u^2 + k_i^2}$ with $\varsigma \in [0, \infty)$. Then, for the TM continuous spectrum we obtain

$$u_{\upsilon,\alpha\alpha}^{(3)}(x) = u_{\upsilon,\alpha\alpha}^{(2)}(x) = 0 \qquad (46)$$
$$u_{\upsilon,x\alpha}^{(3)}(x) = A_7 m_1^{(3)}(x) \qquad (47)$$
$$u_{\upsilon,x\alpha}^{(2)}(x) = A_7 \beta_3 m_1^{(2)}(x) \qquad (48)$$
$$u_{\upsilon,xx}^{(3)}(x) = A_9 m_1^{(3)}(x) \qquad (49)$$
$$u_{\upsilon,xx}^{(2)}(x) = A_9 \beta_3 m_1^{(2)}(x) \qquad (50)$$
$$\overline{\nu}_{\upsilon,\alpha\alpha}^{(3)}(x) = A_8 \frac{(N^2-1)\beta_3}{(N^2\beta_3^2 - \beta_2^2)} m_4^{(3)}(x) \qquad (51)$$
$$\overline{\nu}_{\upsilon,\alpha\alpha}^{(2)}(x) = A_8 \frac{(N^2-1)N^2\beta_3\beta_2}{(N^2\beta_3^2 - \beta_2^2)} m_2^{(2)}(x) \qquad (52)$$
$$\overline{\nu}_{\upsilon,x\alpha}^{(3)}(x) = A_8 m_1^{(3)}(x) \qquad (53)$$
$$\overline{\nu}_{\upsilon,x\alpha}^{(2)}(x) = A_8 N^2 \beta_3 m_1^{(2)}(x) \qquad (54)$$
$$\overline{\nu}_{\upsilon,xx}^{(3)}(x) = A_{10} m_1^{(3)}(x) \qquad (55)$$
$$\overline{\nu}_{\upsilon,xx}^{(2)}(x) = A_{10}\left(N^2\beta_3\right) m_1^{(2)}(x). \qquad (56)$$

The normalization (13) leads to

$$A_9 A_{10} = \left(\pi|\beta_3|\left(\beta_2^2 \sin^2 \beta_2 d + N^4 \beta_3^2 \cos^2 \beta_2 d\right)\right)^{-1} \qquad (57)$$
$$= A_7 A_8. \qquad (58)$$

For the TE continuous spectrum

$$u_{\upsilon,\alpha\alpha}^{(3)}(x) = A_{11} \frac{\left(N^2\beta_3^2 - \beta_2^2\right)}{\beta_3} m_2^{(3)}(x) \qquad (59)$$
$$u_{\upsilon,x\alpha}^{(3)}(x) = A_{11}(N^2-1) m_3^{(3)}(x) \qquad (60)$$
$$u_{\upsilon,\alpha\alpha}^{(2)}(x) = A_{11} \frac{\left(N^2\beta_3^2 - \beta_2^2\right)}{N^2} m_2^{(2)}(x) \qquad (61)$$
$$u_{\upsilon,x\alpha}^{(2)}(x) = -A_{11}(N^2-1)\frac{\beta_2}{N^2} m_1^{(2)}(x) \qquad (62)$$
$$\overline{\nu}_{\upsilon,\alpha\alpha}^{(3)}(x) = A_{12} m_2^{(3)}(x) \qquad (63)$$
$$\overline{\nu}_{\upsilon,\alpha\alpha}^{(2)}(x) = A_{12}\left(N^2\beta_3\right) m_2^{(2)}(x) \qquad (64)$$
$$\overline{\nu}_{\upsilon,x\alpha}^{(3)}(x) = \overline{\nu}_{\upsilon,x\alpha}^{(2)}(x) = 0 \qquad (65)$$

where (13) results in

$$A_{11}A_{12} = \left(\frac{\left(N^2\beta_3^2 - \beta_2^2\right)}{\beta_3}\pi|\beta_3| \times \left(\beta_2^2\cos^2\beta_2 d + \beta_3^2 \sin^2 \beta_2 d\right)\right)^{-1}. \qquad (66)$$

For this geometry (Fig. 2) the dyadic Green's function (26) was compared numerically to the usual Sommerfeld form (see e.g., [23, eqs. (8.191) and (8.192)]), for a few grounded-slab geometries, and the results agreed to within the precision of the numerical integration schemes (i.e., for the Sommerfeld form in [23], and for the continuous spectrum integrals used here).

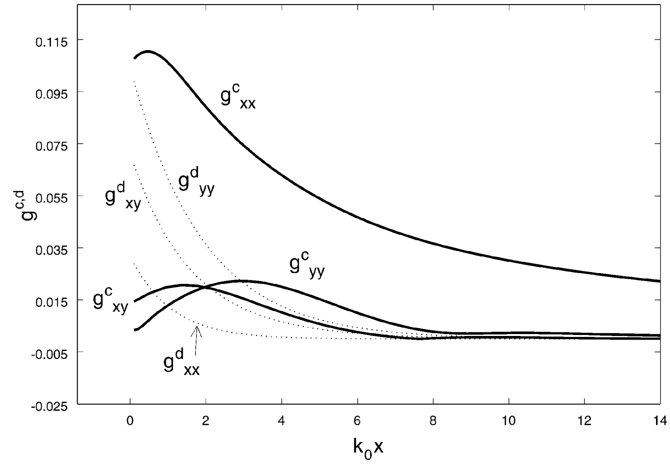

Fig. 3. Magnitude of the continuous (c) and discrete (d) Hertzian potential Green's dyadic components $g_{xx}^{c(d)}$, $g_{yy}^{c(d)}$, and $g_{xy}^{c(d)}$ for a grounded slab (Fig. 2) with $\varepsilon_2 = 2.25\varepsilon_0$, and $d/\lambda_0 = 1/3$. The source is at the origin and the observation point is $(k_0x, k_0y, k_0z) = (k_0x, 3, 0)$.

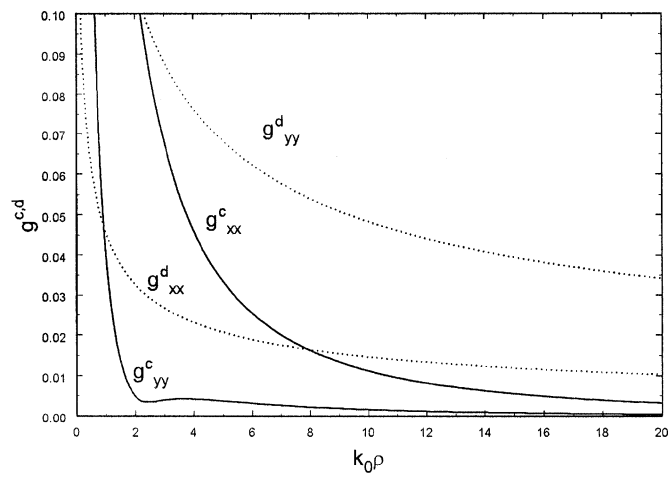

Fig. 4. Magnitude of the continuous (c) and discrete (d) Hertzian potential Green's dyadic components $g_{xx}^{c(d)}$ and $g_{yy}^{c(d)}$ for a grounded slab (Fig. 2) with $\varepsilon_2 = 2.25\varepsilon_0$, $\varepsilon_3 = \varepsilon_0$, and $d/\lambda_0 = 1/3$. The source is at the origin and the observation point $\rho = (y^2 + z^2)^{1/2}$ varies along the interface $(x = 0)$.

To examine the role of the continuous and discrete spectrum, consider a grounded slab as depicted in Fig. 2 having $\varepsilon_2 = 2.25\varepsilon_0$, $\varepsilon_3 = \varepsilon_0$, and $d/\lambda_0 = 1/3$, such that one TE and one TM mode is above cutoff. The source is located at $(x', y', z') = (0, 0, 0)$. Fig. 3 shows the magnitude of the Hertzian potential Green's dyadic components $g_{xx}^{c(d)}$, $g_{yy}^{c(d)}$, and $g_{xy}^{c(d)}$, where the superscript $c\,(d)$ indicates the continuous (discrete) spectrum contributions. The observation point is $(k_0y, k_0z) = (3, 0)$ and the normalized vertical coordinate k_0x varies. As expected, the discrete spectrum component diminishes exponentially away from the air-dielectric interface. For the case considered here, for large distances from the interface the vertical-component continuous spectrum due to a vertical source, g_{xx}^c, dominates.

In Fig. 4, the magnitude of the Hertzian potential Green's dyadic components $g_{xx}^{c(d)}$ and $g_{yy}^{c(d)}$ are shown versus normalized radial distance from the source $(k_0\rho)$ along the air-dielectric interface $(x = 0)$. As expected from the theory of surface waves, the discrete components dominate the continuous spectrum along the interface far from the source. The component $g_{xy}^{c(d)}$ exhibits similar behavior, although it is not radially-symmetric due to the factor $\partial/(\partial y)$ acting on the Hankel function in (20) and (26).

IV. Conclusion

The dyadic Green's function for a layered medium has been constructed using dyadic eigenfunctions of the 3-D Hertzian potential Green's function operator. Properties of the dyadic eigenfunctions were reviewed, and the derivation of the dyadic Green's function was presented. The method represents an elevation to dyadic level of the usual scalar eigenfunction method, and naturally incorporates the coupling of various elements of the Green's dyadic. The method was applied to a grounded dielectric environment, representative of typical printed-circuit antennas, and numerical results were presented showing some typical behavior of the discrete and continuous spectrum constituents.

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