Modeling of Nonlinear, Spatially-Dispersive Plasmas and Semiconductors Under Harmonic Excitation

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***Abstract*—The nonlinear, spatially-dispersive response of a semiconductor or plasma to large-amplitude time-harmonic electromagnetic fields is obtained by solving the nonlinear transport equation using an harmonic expansion. The conduction response, which is nonlinear and generally spatially and temporally dispersive, is given as a hierarchical set of linear second-order differential equations with non-linear forcing terms. The polarization response is assumed linear. A simple slab example is shown that admits analytical solutions for the nonlinear material response to various orders. As the solution order grows, the nonlinear forcing terms grow in complexity, although the differential equations remain second-order. In the static limit, the two lowest-order solutions are shown to identically satisfy the dc transport equation.**

***Index Terms*—Diffusion, nonlinear, plasma, semiconductor.**

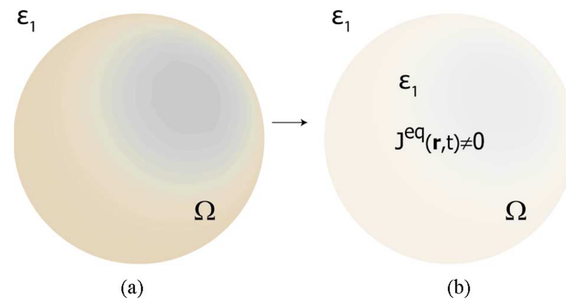

Fig. 1. Left side: general material region Ω containing a nonlinear, spatially and temporally dispersive medium such as a semiconductor or plasma. Right side: equivalent problem where space is homogeneous and equivalent currents are nonzero inside Ω.

I. Introduction

SEMICONDUCTORS and plasmas naturally admit a nonlinear response to strong fields since their charge dynamics are governed by a nonlinear transport equation [1], [2]. Although small input signals engender a linear response, and allow for linearization of the transport equation, the presence of large fields necessitates a more complicated nonlinear analysis [3]. Nonlinear effects in semiconductors and plasmas are too numerous to describe here, although their existence and enhancement are important for controlling the flow of light in optical signal processing [4], [5], self-focusing of light [6], and harmonic generation [7], to name just a few semiconductor applications.

In addition to considering nonlinearities, plasmas and semiconductors exhibit important spatial dispersion effects that often necessitate solving both field and transport equations [8]–[11]. In this work, the interaction of a semiconductor or plasma with a large-amplitude time-harmonic electromagnetic field is examined, where the effects of both spatial dispersion and nonlinearities are included. The method is based on replacing the material with equivalent currents via the volume equivalence principle [13] extended to nonlinear and nonlocal media, and employing a harmonic expansion similar to [14], [15]. This leads to a hierarchical set of linear second-order differential equations with non-linear forcing terms. A simple example is presented to illustrate the method.

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II. Nonlinear Response of a Nonlocal Plasma/Semiconductor

A. General Development

We consider a nonlinear and dispersive material such as a plasma or semiconductor, having extent Ω, immersed in a simple linear and nondispersive dielectric background medium characterized by μ_0, ε_1 and excited by a time-harmonic electromagnetic field $\mathbf{E}^i(\mathbf{r},t)$, $\mathbf{H}^i(\mathbf{r},t)$. Since in what follows the most important nonlinear charge dynamics pertain to mobile changes, for convenience we assume the material has a linear polarization response.

Using the volume equivalence principle, which is well-known for linear, time-harmonic fields [13] and which is easily seen to hold in the time-domain for nonlinear, spatially-dispersive media (as shown in Appendix I), we can remove the material contrast and replace the material in region Ω with the background material, together with equivalent currents

$$\begin{aligned}\mathbf{J}^{\rm eq}(\mathbf{r},t) &= \frac{\partial}{\partial t}(\varepsilon_0-\varepsilon_1)\mathbf{E}(\mathbf{r},t)+\frac{\partial}{\partial t}\mathbf{P}(\mathbf{r},t)+\mathbf{J}^{\rm cond}(\mathbf{r},t)\\ &= \mathbf{J}^{\rm pol}(\mathbf{r},t)+\mathbf{J}^{\rm cond}(\mathbf{r},t) \end{aligned} \quad (1)$$

as depicted in Fig. 1. The current $\mathbf{J}^{\rm cond}$ contains the generally non-linear, spatially- and temporally-dispersive response of charge carriers in region Ω, and $\mathbf{J}^{\rm pol}$ is the linear polarization response (if we assume a simple nondispersive permittivity ε, then $\mathbf{P}=(\varepsilon-\varepsilon_0)\mathbf{E}$ and $\mathbf{J}^{\rm pol}=(\partial/\partial t)(\varepsilon-\varepsilon_1)\mathbf{E}(\mathbf{r},t)$).

For the nonlinear material, we assume semi-classical transport of a single charge species (electrons), although the method is easily extended for multiple types of charges (e.g., holes, ions). We can define a distribution function $f(\mathbf{r},\mathbf{v},t)$ that satisfies Boltzmann's equation, the moments of which lead to conservation of charge (continuity equation) and conservation of momentum (transport equation) [1, Ch. 8]. The resulting continuity and transport equations are

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla \cdot \mathbf{J}^{\mathrm{cond}}(\mathbf{r},t) = 0 \tag{2}$$

and

$$\begin{aligned}\frac{\partial \mathbf{J}^{\mathrm{cond}}(\mathbf{r},t)}{\partial t} &- \frac{e k_B T}{m}\nabla n(\mathbf{r},t) + \left(\mathbf{J}^{\mathrm{cond}}(\mathbf{r},t)\cdot\nabla\right)\mathbf{u} \\ &+ \mathbf{u}\nabla\cdot\mathbf{J}^{\mathrm{cond}}(\mathbf{r},t) - \frac{e^2}{m}n(\mathbf{r},t)\,\mathbf{E}(\mathbf{r},t) \\ &+ \frac{1}{\tau}\mathbf{J}^{\mathrm{cond}}(\mathbf{r},t) = 0\end{aligned} \tag{3}$$

where $e = -q_e$ with q_e being the charge of an electron, $\rho(\mathbf{r},t) = -en(\mathbf{r},t)$ is the charge density, $n(\mathbf{r},t) = \int f(\mathbf{r},\mathbf{v},t)\,d^3v$ is the number density of mobile charge, and $\mathbf{J}^{\mathrm{cond}}(\mathbf{r},t) = -en(\mathbf{r},t)\,\mathbf{u}(\mathbf{r},t) = -e\int \mathbf{v}f(\mathbf{r},\mathbf{v},t)\,d^3v$ is the conduction current density ($\mathbf{u}$ is average velocity). Any generation or recombination mechanisms are included in (2). Note that $\mathbf{E}$ is the self-consistent field (applied field $\mathbf{E}^i$ plus response field $\mathbf{E}^{\mathrm{resp}}$ due to polarization and conduction). In the transport equation (3) we have ignored gravity, and magnetic field effects in the Lorentz force, and we assume the kinetic pressure dyad has been approximated as $p\mathbf{1}$, where $p = nk_BT$ is the scalar kinetic pressure [1, pp. 149–152, 202]; this approximation closes the system of equations, and corresponds to ignoring tangential shear/viscous forces and assuming an isotropic particle velocity distribution. Finally, we have assumed that collisions are represented by a phenomenological relaxation time τ. The term ∇n is associated with spatial dispersion, and the terms $\mathbf{J}^{\mathrm{cond}}\cdot\nabla\mathbf{u}$, $\mathbf{u}\nabla\cdot\mathbf{J}^{\mathrm{cond}}$, and $n\mathbf{E}$ lead to nonlinearities. Neglecting the spatial dispersion and nonlinear current terms and replacing $n(\mathbf{r},t)$ with the equilibrium number density N results in the usual Drude conductivity, which can be used directly in Maxwell's equations (i.e., in this case one does not need to solve the coupled transport-field equations).

Since we are considering the non-linear response due to a large-amplitude time-harmonic input, we assume that the applied field is $\mathbf{E}^i(\mathbf{r},t) = \mathbf{E}^i(\mathbf{r})\,e^{j\omega t} + \mathbf{E}^{i*}(\mathbf{r})\,e^{-j\omega t}$, where the asterisk denotes complex conjugation. We use field expansions similar to those in [14]–[17],

$$\begin{aligned}\mathbf{E}(\mathbf{r},t) &= \mathbf{E}^i(\mathbf{r},t) + \mathbf{E}^s(\mathbf{r},t) \\ &= \mathbf{E}^i(\mathbf{r})\,e^{j\omega t} + \mathbf{E}^{i*}(\mathbf{r})\,e^{-j\omega t} \\ &\quad + \mathbf{E}_0(\mathbf{r}) + \sum_{p=1}^{\infty}\left(\mathbf{E}_p(\mathbf{r})\,e^{jp\omega t} + \mathbf{E}_p^*(\mathbf{r})\,e^{-jp\omega t}\right) \\ \mathbf{J}^{\mathrm{cond}}(\mathbf{r},t) &= \mathbf{J}_0^{\mathrm{cond}}(\mathbf{r}) \\ &\quad + \sum_{p=1}^{\infty}\left(\mathbf{J}_p^{\mathrm{cond}}(\mathbf{r})\,e^{jp\omega t} + \mathbf{J}_p^{\mathrm{cond}*}(\mathbf{r})\,e^{-jp\omega t}\right) \\ n(\mathbf{r},t) &= n_0(\mathbf{r}) + \sum_{p=1}^{\infty}\left(n_p(\mathbf{r})\,e^{jp\omega t} + n_p^*(\mathbf{r})\,e^{-jp\omega t}\right) \\ \mathbf{u}(\mathbf{r},t) &= \mathbf{u}_0(\mathbf{r}) + \sum_{p=1}^{\infty}\left(\mathbf{u}_p(\mathbf{r})\,e^{jp\omega t} + \mathbf{u}_p^*(\mathbf{r})\,e^{-jp\omega t}\right)\end{aligned} \tag{4}$$

where $\mathbf{E}^s$ is the scattered (material response) field, in both Maxwell's equations, the transport and continuity equations, and in the definition for current density, $\mathbf{J}^{\mathrm{cond}} = qn\mathbf{u}$. Note that the order zero quantities are equilibrium values, and so $\mathbf{u}_0(\mathbf{r}) = \mathbf{J}_0^{\mathrm{cond}}(\mathbf{r}) = \mathbf{0}$, and $n_0(\mathbf{r}) = N$ (for a plasma N is the equilibrium density of electrons, i.e., the density of ionized atoms, and for a semiconductor N is the doping density).

Inserting the expansions into the transport equation, multiplying out terms and using the linear independence of $e^{j\alpha t}$, $\alpha = 0, \pm\omega, \pm 2\omega, \ldots$, we can equate like frequency terms. As an illustrative example, keeping up to second-order terms for convenience, for e^0 we obtain the zeroth-order transport equation

$$\begin{aligned}&\mathbf{J}_1^{*\mathrm{cond}}\cdot\nabla\mathbf{u}_1 + \mathbf{J}_1^{\mathrm{cond}}\cdot\nabla\mathbf{u}_1^* \\ &\quad + \mathbf{J}_2^{*\mathrm{cond}}\cdot\nabla\mathbf{u}_2 + \mathbf{J}_2^{\mathrm{cond}}\cdot\nabla\mathbf{u}_2^* \\ &\quad + \mathbf{u}_1^*\nabla\cdot\mathbf{J}_1^{\mathrm{cond}} + \mathbf{u}_1\nabla\cdot\mathbf{J}_1^{*\mathrm{cond}} \\ &\quad + \mathbf{u}_2^*\nabla\cdot\mathbf{J}_2^{\mathrm{cond}} + \mathbf{u}_2\nabla\cdot\mathbf{J}_2^{*\mathrm{cond}} - \frac{e^2}{m}(N\mathbf{E}_0) \\ &\quad - \frac{e^2}{m}\left(n_1^*\left(\mathbf{E}^i + \mathbf{E}_1\right)\right) - \frac{e^2}{m}\left(n_1\left(\mathbf{E}^{i*} + \mathbf{E}_1^*\right)\right) \\ &\quad - \frac{e^2}{m}(n_2^*\mathbf{E}_2) - \frac{e^2}{m}(n_2\mathbf{E}_2^*) = 0\end{aligned} \tag{5}$$

and for $e^{j\omega t}$ we obtain the first-order transport equation

$$\begin{aligned}&\left(j\omega + \frac{1}{\tau}\right)\mathbf{J}_1^{\mathrm{cond}} - \frac{ek_BT}{m}\nabla n_1 \\ &\quad + \mathbf{J}_2^{\mathrm{cond}}\cdot\nabla\mathbf{u}_1^* + \mathbf{J}_1^{*\mathrm{cond}}\cdot\nabla\mathbf{u}_2 \\ &\quad + \mathbf{u}_2\nabla\cdot\mathbf{J}_1^{*\mathrm{cond}} + \mathbf{u}_1^*\nabla\cdot\mathbf{J}_2^{\mathrm{cond}} - \frac{e^2}{m}(n_1\mathbf{E}_0) \\ &\quad - \frac{e^2}{m}\left(N\left(\mathbf{E}^i + \mathbf{E}_1\right)\right) - \frac{e^2}{m}\left(n_2\left(\mathbf{E}^{i*} + \mathbf{E}_1^*\right)\right) \\ &\quad - \frac{e^2}{m}n_1^*\mathbf{E}_2 = 0.\end{aligned} \tag{6}$$

Higher-order transport equations are obtained analogously, and we obtain a similar set of equations for negative frequencies and conjugate variables. For example, for $e^{-j\omega t}$,

$$\begin{aligned}&\left(-j\omega + \frac{1}{\tau}\right)\mathbf{J}_1^{*\mathrm{cond}} - \frac{ek_BT}{m}\nabla n_1^* + \mathbf{J}_2^{*\mathrm{cond}}\cdot\nabla\mathbf{u}_1 \\ &+ \mathbf{J}_1^{\mathrm{cond}}\cdot\nabla\mathbf{u}_2^* + \mathbf{u}_2^*\nabla\cdot\mathbf{J}_1^{\mathrm{cond}} + \mathbf{u}_1\nabla\cdot\mathbf{J}_2^{*\mathrm{cond}} - \frac{e^2}{m}n_1^*\mathbf{E}_0 \\ &- \frac{e^2}{m}n_2^*\left(\mathbf{E}^i + \mathbf{E}_1\right) - \frac{e^2}{m}N\left(\mathbf{E}^{i*} + \mathbf{E}_1^*\right) - \frac{e^2}{m}n_1\mathbf{E}_2^* = 0.\end{aligned} \tag{7}$$

We assume that for $\varkappa$ representing any of the expanded variables,

$$\varkappa(\mathbf{r},t) = \varkappa_0 + \sum_{p=1}^{\infty}\left(\varkappa_p(\mathbf{r})\,e^{jp\omega t} + \varkappa_p^*(\mathbf{r})\,e^{-jp\omega t}\right) \tag{8}$$

we have

$$\|\varkappa_p\| \ll \|\varkappa_{p-1}\| \tag{9}$$

using an appropriate norm $\|\cdot\|$, so that the given expansions converge. Notice that in the pth order transport equation there are pth order terms, $(p+2)$nd order terms, $(p+4)$th order terms, etc., if all higher-order expansions were included. Furthermore, the higher-order terms are always associated with conjugate variables (i.e., down-shifted negative frequency terms). In our specific case, where we included terms in the expansion up to second order, the 0th order equation ($\omega = 0$) has 0th (e.g., $N\mathbf{E}_0$), 2nd (e.g., $\mathbf{J}_1^{\rm cond}\cdot\nabla\mathbf{u}_1^*$), and 4th order (e.g., $\mathbf{J}_2^{\rm cond}\cdot\nabla\mathbf{u}_2^*$) terms, the 1st order equation (ω) has 1st (e.g., $\mathbf{E}_1$) and 3rd order (e.g., $\mathbf{J}_2^{\rm cond}\cdot\nabla\mathbf{u}_1^*$) terms, and the 2nd order equation has only second order terms (due to the number of terms retained).

Although in general we have an infinite-dimensional system, at this point we make two approximations. First, we truncate the series at some finite M, and second, using (9) we ignore the terms of order higher than p in the pth order equation (e.g., we ignore second-order terms in the first order equation). Since for the pth order equation these terms are of order $(p+2)$, if (9) is true (as is the case for the example given below) then this should be an excellent approximation. These higher-order terms are always associated with conjugate variables, such that dropping the $(p+2)$ nd and higher-order terms in the pth order equation is equivalent to using the expansions (4) without the conjugate terms.

From $\mathbf{J}^{\rm cond} = qn\mathbf{u}$ this leads to

$$\mathbf{J}_p^{\rm cond}(\mathbf{r}) = -e\sum_{n=0}^{M}\sum_{m=1}^{M}\delta_{(n+m),p}\, n_n\mathbf{u}_m \tag{10}$$

where $\delta_{n,m}$ is the Kronecker delta function. We also have the continuity equation

$$\nabla\cdot\mathbf{J}^{\rm cond} = -\frac{\partial\rho^{\rm cond}}{\partial t} = -e\frac{\partial(N-n(\mathbf{r},t))}{\partial t} = e\frac{\partial n(\mathbf{r},t)}{\partial t} \tag{11}$$

leading to[1]

$$\nabla\cdot\mathbf{J}_p^{\rm cond}(\mathbf{r}) = ej(p\omega)\,n_p(\mathbf{r}). \tag{12}$$

and so

$$\nabla n_p(\mathbf{r}) = \frac{\nabla\nabla\cdot\mathbf{J}_p^{\rm cond}(\mathbf{r})}{ej(p\omega)}. \tag{13}$$

For the polarization response assuming a similar harmonic expansion for $\mathbf{P}$, we have

$$\mathbf{J}_p^{\rm pol}(\mathbf{r}) = j(p\omega)(\varepsilon_0-\varepsilon_1)\left(\delta_{p,1}\mathbf{E}^i(\mathbf{r}) + \mathbf{E}_p(\mathbf{r})\right) + j(p\omega)\mathbf{P}_p(\mathbf{r}). \tag{14}$$

The transport equation for $p = 0$ is

$$-\frac{e^2}{m}N\mathbf{E}_0(\mathbf{r}) = 0 \rightarrow \mathbf{E}_0 = 0. \tag{15}$$

Using the continuity equation (12)–(13), the pth order $(p > 0)$ transport equation is[2]

$$\left(1-\frac{D_p}{j(p\omega)}\nabla\nabla\cdot\right)\mathbf{J}_p^{\rm cond}(\mathbf{r}) - \sigma_p\mathbf{E}_p(\mathbf{r}) = \frac{\tau}{1+j(p\omega)\tau}\mathbf{S}_p \tag{16}$$

[1]Note that here and in the following $j(p\omega)$ is simply $jp\omega$; it does not represent a functional dependence.

[2]While this paper was in review [17] appeared, wherein [17, (2)–(3)] are the same as (16) for $p = 1, 2$.

where $\mathbf{E}_p$ is the electric field due to $\left(\mathbf{J}_p^{\rm cond} + \mathbf{J}_p^{\rm pol}\right)$,

$$\begin{aligned}\sigma_p(\omega) &= \frac{Ne^2\tau}{m(1+j(p\omega)\tau)} = \frac{\sigma_0}{1+j(p\omega)\tau}\\ D_p(\omega) &= \frac{\beta\tau}{1+j(p\omega)\tau} = \frac{D_0}{1+j(p\omega)\tau}\\ \beta &= \frac{k_BT}{m}\end{aligned} \tag{17}$$

are the conductivity and diffusion coefficients evaluated at frequency $p\omega$, and where $\mathbf{S}_p$ are generally pth order nonlinear (except for $p = 1$ which is linear) source terms involving lower-order quantities,

$$\begin{aligned}\mathbf{S}_p = &-\sum_{n=1}^{M}\sum_{m=1}^{M}\delta_{(n+m),p}\\ &\times\left(\mathbf{J}_n^{\rm cond}\cdot\nabla\mathbf{u}_m + \mathbf{u}_n\nabla\cdot\mathbf{J}_m^{\rm cond} - \frac{e^2}{m}n_n\mathbf{E}_m\right)\\ &+\frac{e^2}{m}n_{p-1}\mathbf{E}^i\end{aligned} \tag{18}$$

where

$$\begin{aligned}\mathbf{u}_p &= \frac{1}{N}\left(\frac{1}{q}\mathbf{J}_p^{\rm cond} - \sum_{n=1}^{M}\sum_{m=1}^{M}\delta_{(n+m),p}n_n\mathbf{u}_m\right)\\ &= \frac{1}{N}\left(\frac{1}{q}\mathbf{J}_p^{\rm cond} - \sum_{n=1}^{M}\sum_{m=1}^{M}\delta_{(n+m),p}\frac{\nabla\cdot\mathbf{J}_n^{\rm cond}}{ej(n\omega)}\mathbf{u}_m\right).\end{aligned} \tag{19}$$

As an example,

$$\begin{aligned}\mathbf{S}_1 &= \left(\frac{Ne^2}{m}\right)\mathbf{E}^i(\mathbf{r})\\ \mathbf{S}_2 &= -\mathbf{J}_1^{\rm cond}\cdot\nabla\mathbf{u}_1 - \mathbf{u}_1\nabla\cdot\mathbf{J}_1^{\rm cond} + \frac{e^2}{m}n_1\left(\mathbf{E}_1+\mathbf{E}^i\right)\\ \mathbf{S}_3 &= -\mathbf{J}_2^{\rm cond}\cdot\nabla\mathbf{u}_1 - \mathbf{J}_1^{\rm cond}\cdot\nabla\mathbf{u}_2 - \mathbf{u}_2\nabla\cdot\mathbf{J}_1^{\rm cond}\\ &\quad - \mathbf{u}_1\nabla\cdot\mathbf{J}_2^{\rm cond} + \frac{e^2}{m}n_1\mathbf{E}_2 + \frac{e^2}{m}n_2\left(\mathbf{E}_1+\mathbf{E}^i\right)\end{aligned} \tag{20}$$

and

$$\begin{aligned}\mathbf{u}_1 &= \frac{1}{qN}\mathbf{J}_1^{\rm cond}\\ \mathbf{u}_2 &= \frac{1}{N}\left(\frac{1}{q}\mathbf{J}_2^{\rm cond} - \frac{\nabla\cdot\mathbf{J}_1^{\rm cond}}{ej(\omega)}\mathbf{u}_1\right)\\ \mathbf{u}_3 &= \frac{1}{N}\left(\frac{1}{q}\mathbf{J}_3^{\rm cond} - \frac{\nabla\cdot\mathbf{J}_2^{\rm cond}}{ej(2\omega)}\mathbf{u}_1 - \frac{\nabla\cdot\mathbf{J}_1^{\rm cond}}{ej(\omega)}\mathbf{u}_2\right).\end{aligned} \tag{21}$$

It is easy to see that $|\mathbf{J}_p| \propto \left|\mathbf{E}^i\right|^p$.

Although we have a nonlinear transport equation, from linear Maxwell's equations the pth-order field $\mathbf{E}_p(\mathbf{r})$ is linear in the pth order currents $\mathbf{J}_p^{\rm cond}(\mathbf{r})$ and $\mathbf{J}_p^{\rm pol}(\mathbf{r})$; this is the usual frequency-domain response [13] (at frequency $(p\omega)$) due to the pth order currents,

$$\begin{aligned}\mathbf{E}_p(\mathbf{r}) = \frac{\left(k_{1,p}^2+\nabla\nabla\cdot\right)}{j(p\omega)\varepsilon_1}\int_\Omega g_p(\mathbf{r},\mathbf{r}')\\ \times\left(\mathbf{J}_p^{\rm cond}(\mathbf{r}') + \mathbf{J}_p^{\rm pol}(\mathbf{r}')\right)dV'\end{aligned} \tag{22}$$

where $k_{1,p}^2 = (p\omega)^2 \mu_0 \varepsilon_1$ and the Green's function is $g_p = \exp(-jk_{1,p}R)/4\pi R$, $R = \mathbf{r} - \mathbf{r}'$. This linear association of $\mathbf{E}_p$ with $\mathbf{J}_p$ comes from the following. We insert the expansions (4) into Maxwell's equations (MEs) for the scattered field (68),

$$\nabla \times \mathbf{E}^s(\mathbf{r}, t) = -\frac{\partial}{\partial t}\mu_0 \mathbf{H}^s(\mathbf{r}, t)$$
$$\nabla \times \mathbf{H}^s(\mathbf{r}, t) = \frac{\partial}{\partial t}\varepsilon_1 \mathbf{E}^s(\mathbf{r}, t) + \mathbf{J}^{\mathrm{eq}}(\mathbf{r}, t) \tag{23}$$

along with a similar expansion for $\mathbf{H}(\mathbf{r}, t)$. Exploiting linear independence of $e^{\pm jp\omega t}$ leads to the equilibrium MEs

$$\nabla \times \mathbf{E}_0(\mathbf{r}) = 0$$
$$\nabla \times \mathbf{H}_0(\mathbf{r}) = \mathbf{J}_0^{\mathrm{eq}}(\mathbf{r}) \tag{24}$$

the first order MEs

$$\nabla \times \mathbf{E}_1(\mathbf{r}) = -j\omega\mu_0 \mathbf{H}_1(\mathbf{r})$$
$$\nabla \times \mathbf{H}_1(\mathbf{r}) = j\omega\varepsilon_1 \mathbf{E}_1(\mathbf{r}) + \mathbf{J}_1^{\mathrm{eq}}(\mathbf{r}) \tag{25}$$

the second order MEs

$$\nabla \times \mathbf{E}_2(\mathbf{r}) = -j2\omega\mu_0 \mathbf{H}_2(\mathbf{r})$$
$$\nabla \times \mathbf{H}_2(\mathbf{r}) = j2\omega\varepsilon_1 \mathbf{E}_2(\mathbf{r}) + \mathbf{J}_2^{\mathrm{eq}}(\mathbf{r}) \tag{26}$$

and so on. Thus, we can interpret the field $\mathbf{E}_0$ as the $\omega = 0$ field due to $\mathbf{J}_0^{\mathrm{eq}}$, the field $\mathbf{E}_1$ as the field at ω due to the response current $\mathbf{J}_1^{\mathrm{eq}}$ (oscillating at ω), the field $\mathbf{E}_2$ as the field at 2ω associated with $\mathbf{J}_2^{\mathrm{eq}}$ at 2ω, and similarly for higher orders $\mathbf{E}_p$ (and also negative frequencies, although we have already dropped these terms). Thus, the Maxwell equation subsystem is linear in the various orders of current. A similar method involving harmonic expansions, leading to sets of harmonic Maxwell's equations and Green's functions for frequencies $(p\omega)$ are considered for nonlinear magnetic problems in [18], [19]. If one tries to forgo the transport equation and treat the nonlinearity as a constitutive relation embedded in Maxwell's equations, then one cannot generally use the concept of Green's functions.

The final system of equations to be solved is the coupled system of integral-differential (14) and (16),

$$\left(1 - \frac{D_p}{j(p\omega)}\nabla\nabla\cdot\right)\mathbf{J}_p^{\mathrm{cond}}(\mathbf{r}) - \left\{\frac{\sigma_p(k_{1,p}^2 + \nabla\nabla\cdot)}{j(p\omega)\varepsilon_1}\int_\Omega g_p(\mathbf{r}, \mathbf{r}')\left(\mathbf{J}_p^{\mathrm{cond}}(\mathbf{r}') + \mathbf{J}_p^{\mathrm{pol}}(\mathbf{r}')\right)dV'\right\} = \frac{\tau}{1 + j(p\omega)\tau}\mathbf{S}_p \tag{27}$$

and

$$\mathbf{J}_p^{\mathrm{pol}}(\mathbf{r}) = j(p\omega)(\varepsilon_0 - \varepsilon_1)\left(\delta_{p,1}\mathbf{E}^i(\mathbf{r}) + \frac{(k_{1,p}^2 + \nabla\nabla\cdot)}{j(p\omega)\varepsilon_1}\int_\Omega g_p(\mathbf{r}, \mathbf{r}')\left(\mathbf{J}_p^{\mathrm{cond}}(\mathbf{r}') + \mathbf{J}_p^{\mathrm{pol}}(\mathbf{r}')\right)dV'\right)$$
$$j(p\omega)\mathbf{P}_p(\mathbf{r}) \tag{28}$$

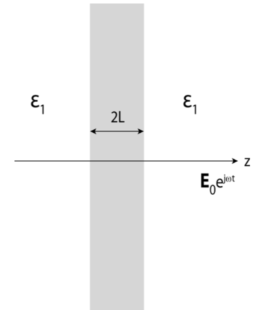

Fig. 2. Semi-infinite slab having width $2L$ consisting of nonlinear, dispersive material.

for $p > 0$. In general, we need to know the relationship between $\mathbf{P}_p$ and $\mathbf{E}_p$. In the special case of a nondispersive permittivity ε, $\mathbf{P}_p = (\varepsilon - \varepsilon_0)\left(\delta_{p,1}\mathbf{E}^i(\mathbf{r}) + \mathbf{E}_p(\mathbf{r})\right)$, and (28) reduces to

$$\mathbf{J}_p^{\mathrm{pol}}(\mathbf{r}) = j(p\omega)(\varepsilon - \varepsilon_1)\left(\delta_{p,1}\mathbf{E}^i(\mathbf{r}) + \frac{(k_{1,p}^2 + \nabla\nabla\cdot)}{j(p\omega)\varepsilon_1}\int_\Omega g_p(\mathbf{r}, \mathbf{r}')\left(\mathbf{J}_p^{\mathrm{cond}}(\mathbf{r}') + \mathbf{J}_p^{\mathrm{pol}}(\mathbf{r}')\right)dV'\right). \tag{29}$$

The system (27)–(28) can be solved starting with $p = 1$, for which we only need to know $\mathbf{E}^i$. Although $\mathbf{J}^{\mathrm{cond}}$ appears in the equations, since charge number density n and velocity $\mathbf{u}$ are present in $\mathbf{S}$, and since $\mathbf{J}^{\mathrm{cond}} = -en\mathbf{u}$, the system (27)–(28) can be solved for the unknowns $\left(\mathbf{J}_1^{\mathrm{pol}}, n_1, \mathbf{u}_1\right)$ using, e.g., the method of moments (alternatively, one can eliminate $\mathbf{u}$ from the equations and work only with $\mathbf{J}$). Once these quantities are obtained for $p = 1$ we can solve the second-order equation using these (now known) quantities in $\mathbf{S}_2$, and so $\mathbf{J}_2^{\mathrm{cond}}$ and $\mathbf{J}_2^{\mathrm{pol}}$ can be determined, and so on for higher-order equations. That is, we start with $p = 1$ and successively solve for higher orders $p = 2, 3, \ldots, M$. For simple problems like the example shown below, this leads to analytical solutions for the currents.

If we ignore spatial dispersion $(D_p = 0)$ then (27) and (29) can be combined upon defining the usual complex permittivity that includes conductivity, resulting in a single volume/domain integral equation identical to the usual dielectric scatterer or resonator case, except for the nonlinear source term $\mathbf{S}_p$ (and for $p = 1$ this is the usual linear source term).

III. One-Dimensional Example

As an illustrative example of the described method that leads to an analytical solution, assume a laterally-infinite semiconductor/plasma extends from $z = -L$ to $z = L$, as depicted in Fig. 2. The excitation is $\mathbf{E}^i = \hat{\mathbf{z}}E_0 e^{j\omega t}$, and so all vector quantities are now in the z direction, and the only spatial variation is z. This geometry was considered in [11], where the linear response was evaluated (see also [12] where a wire-medium slab was considered). For simplicity, we assume a nondispersive permittivity ε.

In one-dimension, the Green's function is

$$g(z, z') = \frac{e^{-jk_1|z - z'|}}{2jk_1} \tag{30}$$

and, using Leibnitz's theorem [20], if $z \in (-L, L)$

$$\left(k_1^2 + \frac{d^2}{dz^2}\right) \int_{-L}^{L} \frac{e^{-jk_1|z-z'|}}{2jk_1} J(z')dz' = -J(z) \tag{31}$$

the scattered field is

$$\begin{aligned} E^s(z) &= \frac{1}{j\omega\varepsilon_1}\left(k_1^2 + \frac{d^2}{dz^2}\right) \\ &\quad \times \int_{-L}^{L} \frac{e^{-jk_1|z-z'|}}{2jk_1} \left(J^{\text{pol}}(z') + J^{\text{cond}}(z')\right) dz' \\ &= -\frac{1}{j\omega\varepsilon_1}\left(J^{\text{pol}}(z) + J^{\text{cond}}(z)\right). \end{aligned} \tag{32}$$

(the same thing happens in the static case for $g(z, z') = -|z - z'|/2.$). Note also that for $z \notin (-L, L)$

$$\left(k_1^2 + \frac{d^2}{dz^2}\right) \int_{-L}^{L} \frac{e^{-jk_1(z-z')}}{2jk_1} J(z')dz' = 0 \tag{33}$$

and so the slab does not scatter a field into the external space. We first consider the method of solution starting with $p = 1$ and progressing to higher orders.

1) First-Order Response: From (16) for $p = 1$, using

$$E_1(z) = -\frac{J_1^{\text{cond}}(z) + J_1^{\text{pol}}(z)}{j\omega\varepsilon_1} \tag{34}$$

we have the coupled linear equations

$$\begin{aligned} J_1^{\text{pol}} = &-(j\omega[\varepsilon - \varepsilon_1])\frac{1}{j\omega\varepsilon_1}\left(J_1^{\text{pol}} + J_1^{\text{cond}}\right) \\ &+ (j\omega[\varepsilon - \varepsilon_1]) E^i(z)\left(1 - \frac{D(\omega)}{j\omega}\frac{d^2}{dz^2}\right) J_1^{\text{cond}}(z) \\ &+ \frac{\sigma_1(\omega)}{j\omega\varepsilon_1}\left(J_1^{\text{cond}}(z) + J_1^{\text{pol}}(z)\right) \\ = &\,\sigma_1(\omega) E^i(z). \end{aligned} \tag{35}$$

These can easily be decoupled such that the polarization equation becomes

$$J_1^{\text{pol}} = -\frac{[\varepsilon - \varepsilon_1]}{\varepsilon} J_1^{\text{cond}} + \frac{j\omega\varepsilon_1[\varepsilon - \varepsilon_1]}{\varepsilon} E^i \tag{36}$$

and the conduction equation is

$$\left(1 - \frac{1}{k_{D1}^2}\frac{d^2}{dz^2}\right) J_1^{\text{cond}}(z) = \frac{j\omega\varepsilon_1\sigma_1(\omega)}{j\omega\varepsilon + \sigma_1(\omega)} E^i(z) \tag{37}$$

where

$$k_{D1} = \sqrt{\frac{j\omega\varepsilon + \sigma_1}{D_1\varepsilon}} \tag{38}$$

is a Debye wavenumber [21], [22].

For the conduction equation, the homogeneous solution is

$$J_1^{\text{cond}} = C_1 e^{k_{D1}z} + C_2 e^{-k_{D1}z} \tag{39}$$

and the particular solution is

$$J_1^{\text{cond}} = \frac{j\omega\varepsilon_1\sigma_1}{j\omega\varepsilon + \sigma_1} E^i \tag{40}$$

for E^i being constant in space. Enforcing $J_1^{\text{cond}}(\pm L) = 0$ [9], the solution is

$$J_1^{\text{cond}}(z) = \frac{j\omega\varepsilon_1\sigma_1}{j\omega\varepsilon + \sigma_1}\left(1 - \frac{\cosh(k_{D1}z)}{\cosh(k_{D1}L)}\right) E^i. \tag{41}$$

Therefore, we know $u_1(z) = J_1^{\text{cond}}(z)/N$ and $n_1(z) = (d/dz)J_1^{\text{cond}}(z)/ej\omega$. The associated charge is given as

$$\rho_1^{\text{cond}}(z) = \frac{k_{D1}\varepsilon_1\sigma_1}{j\omega\varepsilon + \sigma_1}\frac{\sinh(k_{D1}z)}{\cosh(k_{D1}L)} E^i. \tag{42}$$

This first-order response agrees with the results of the linear analysis in [11].

2) Second-Order Response: Since

$$E_2(z) = -\frac{J_2^{\text{cond}}(z) + J_2^{\text{pol}}(z)}{j2\omega\varepsilon_1} \tag{43}$$

and using (34), we obtain

$$J_2^{\text{pol}}(z) = -\frac{[\varepsilon - \varepsilon_1]}{\varepsilon} J_2^{\text{cond}}(z) \tag{44}$$

such that $E_2(z) = -J_2^{\text{cond}}/j2\omega\varepsilon$. The second order conduction equation becomes

$$\left(1 - \frac{1}{k_{D2}^2}\frac{d^2}{dz^2}\right) J_2^{\text{cond}}(z) = \frac{\tau j2\omega\varepsilon}{(j2\omega\varepsilon + \sigma_2)(1 + j2\omega\tau)} S_2 \tag{45}$$

where

$$k_{D2} = \sqrt{\frac{j2\omega\varepsilon + \sigma_2}{D_2\varepsilon}} \tag{46}$$

is a Debye wavenumber associated with second-order effects, and where

$$S_2 = \left\{\left(\frac{2}{eN} + \frac{e}{m\varepsilon\omega^2}\right) J_1^{\text{cond}} + \frac{e}{mj\omega}\frac{\varepsilon_1}{\varepsilon} E^i\right\} \frac{d}{dz} J_1^{\text{cond}}. \tag{47}$$

The homogeneous solution is

$$J_2^{\text{cond,hom}} = C_1 e^{k_{D2}z} + C_2 e^{-k_{D2}z} \tag{48}$$

and, using the solution for J_1^{cond}, the particular solution is

$$\begin{aligned} J_2^{\text{cond,part}}(z) = &\, a_1 + a_2 \sinh(k_{D1}z) + a_3 \cosh(k_{D1}z) \\ &+ a_4 \sinh(2k_{D1}z) + a_5 \cosh(2k_{D1}z). \end{aligned} \tag{49}$$

Evaluating the constants and enforcing the same boundary conditions as for the first-order response, the second-order solution is

$$\begin{aligned} J_2^{\text{cond}} = &\, a_2 \sinh(k_{D1}L)\left(\frac{\sinh(k_{D1}z)}{\sinh(k_{D1}L)} - \frac{\sinh(k_{D2}z)}{\sinh(k_{D2}L)}\right) \\ &+ a_4 \sinh(2Lk_{D1})\left(\frac{\sinh(2k_{D1}z)}{\sinh(2Lk_{D1})} - \frac{\sinh(k_{D2}z)}{\sinh(k_{D2}L)}\right) \end{aligned} \tag{50}$$

and for the associated charge,

$$\begin{aligned} &-j2\omega\rho_2^{\text{cond}} \\ &= a_2 \sinh(k_{D1}L)\left(\frac{k_{D1}\cosh(k_{D1}z)}{\sinh(k_{D1}L)} - \frac{k_{D2}\cosh(k_{D2}z)}{\sinh(k_{D2}L)}\right) \\ &+ a_4 \sinh(2Lk_{D1})\left(\frac{2k_{D1}\cosh(2k_{D1}z)}{\sinh(2Lk_{D1})} - \frac{k_{D2}\cosh(k_{D2}z)}{\sinh(k_{D2}L)}\right) \end{aligned} \tag{51}$$

where

$$a_2 = \frac{k_{D2}^2}{k_{D2}^2 - k_{D1}^2} \frac{A(B+C)}{(j2\omega\varepsilon + \sigma_2)\cosh(k_{D1}L)} \left(E^i\right)^2$$
$$a_4 = -\frac{k_{D2}^2}{k_{D2}^2 - 4k_{D1}^2} \frac{AC}{(j2\omega\varepsilon + \sigma_2)\, 2\cosh^2(k_{D1}L)} \left(E^i\right)^2$$
$$A = -Fk_{D1}\frac{j2\omega\tau\varepsilon}{1+j2\omega\tau},\ B = \frac{\varepsilon_1 e}{\varepsilon m j\omega}$$
$$C = \left(\frac{2}{eN} + \frac{e}{m\varepsilon\omega^2}\right)F, \quad F = \frac{j\omega\varepsilon_1\sigma}{j\omega\varepsilon + \sigma}. \quad (52)$$

Details for the $p = 3$ and higher-order cases will be omitted, but can be obtained from the pth order transport equation

$$\left(1 - \frac{1}{k_{Dp}^2}\frac{d^2}{dz^2}\right) J_p^{\text{cond}}(z) = R_p \quad (53)$$

where R_p are nonlinear source terms involving lower-order solutions and the incident field, and where

$$k_{Dp} = \sqrt{\frac{j(p\omega)\varepsilon + \sigma_p}{D_p\varepsilon}} \quad (54)$$

is a pth order Debye constant.

3) Quasi-Static and Static Cases: From the above results, simpler expressions in the low-frequency limit $(p\omega\tau \ll 1, p\omega\varepsilon \ll \sigma)$ are easily obtained,

$$\sigma_p(\omega) = \frac{Ne^2\tau}{m(1+j(p\omega)\tau)} = \frac{\sigma_0}{1+j(p\omega)\tau} \rightarrow \sigma_0 \quad (55)$$
$$D_p(\omega) \rightarrow D_0, k_{Dp} \rightarrow \sqrt{\frac{\sigma_0}{D_0\varepsilon}} = \sqrt{\frac{Ne^2}{k_BT\varepsilon}} = k_D \quad (56)$$

and

$$a_2 = \frac{4\omega^2\varepsilon_1^2 k_D\tau}{eN\cosh(k_DL)}\left(E^i\right)^2$$
$$a_4 = \frac{j\omega e\varepsilon_1^2 k_D\tau}{3m\sigma_0\cosh^2(k_DL)}\left(E^i\right)^2. \quad (57)$$

In the quasi-static regime the expressions for charge are

$$\rho_1^{\text{cond}}(z) = k_D\varepsilon_1\frac{\sinh(k_Dz)}{\cosh(k_DL)}E^i$$
$$\rho_2^{\text{cond}}(z) = -b_4k_D(\cosh(2k_Dz) - \cosh(k_DL)\cosh(k_Dz)) \quad (58)$$

where $b_4 = a_4/j\omega$ is proportional to $\left(E^i\right)^2$.

As $\omega \rightarrow 0$, $J_{1,2}^{\text{cond}} \rightarrow 0$ although $\rho_{1,2}^{\text{cond}}$ converge to finite values (their small ω limits),

$$\rho_{1+2}^{\text{cond}}(z) = \varepsilon_1 k_D\frac{\sinh(k_Dz)}{\cosh(kL)}E^i - \frac{\varepsilon_1^2k_D^2}{3Ne}\left(\frac{\cosh(2k_Dz) - \cosh(k_DL)\cosh(k_Dz)}{\cosh^2(k_DL)}\right)\left(E^i\right)^2. \quad (59)$$

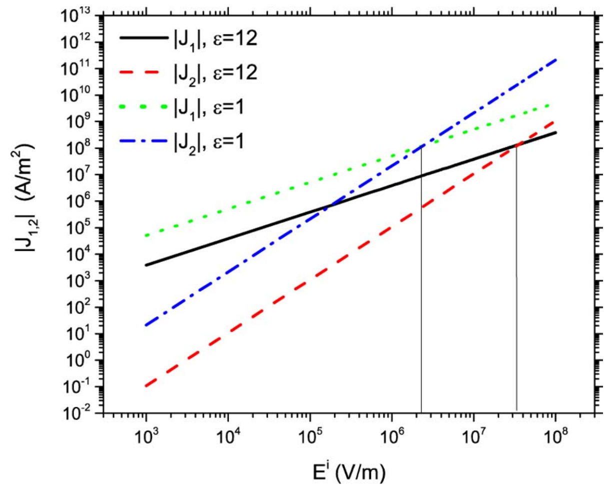

Fig. 3. Magnitude of the first two orders of current vs. strength of the applied electric field E^i (V/m) at $z = -L/3$ for $\varepsilon_h = 1$ and $\varepsilon_h = 12$, and doping $N = 10^{22}\ \text{m}^{-3}$.

It is easy to show that (59) satisfies the DC transport equation to second order, as shown in Appendix II.

A. Numerical Results

As a numerical example, we assume a semiconducting slab having total thickness $2L = 100$ nm, $f = 1$ THz, $m_{eff} = m_e$, and $\tau = 2.156 \times 10^{-13}$ s. Fig. 3 shows the magnitude of the first two orders of current vs. strength of the applied electric field E^i (V/m) at $z = -L/3$ for $\varepsilon_h = 1$ and $\varepsilon_h = 12$, and doping $N = 10^{22}\ \text{m}^{-3}$. The vertical lines denote the value of incident field where the $\left|\mathbf{J}_1^{\text{cond}}\right| = \left|\mathbf{J}_2^{\text{cond}}\right|$; far below this value only the first-order current is important. For $\varepsilon_h = 1$, second-order effects become important at a somewhat lower value of applied field than for $\varepsilon_h = 12$ (for this example, as shown in Fig. 4, nonlinear effects become important for $E^i > 3 \times 10^5$ V/m).

The value of E^i at which the first two solutions cross and second-order effects become dominant depends on the location within the slab, and material, energy, and size parameters ε, k_BT, N, τ, and L. An analytical formula for the crossing point can be obtained by equating J_1^{cond} and J_2^{cond}, but the resulting expression is complicated and will be omitted here. However, a simple expression for the crossing point for charge density is obtained by equating ρ_1^{cond} and ρ_2^{cond},

$$E^i_{\rho_1=\rho_2} = \frac{(Nk_BT\varepsilon)^{1/2}}{\varepsilon_1} \times \frac{3\cosh(k_DL)\sinh(k_Dz)}{\cosh(k_Dz)\cosh(k_DL) - \cosh(2k_Dz)}. \quad (60)$$

Increasing the layer permittivity shifts the crossing point to higher field values, as also seen in Fig. 3 for the currents.

Fig. 4 shows the current density versus position at $t = 2$ ps, $\varepsilon_h = 1$, and doping density $N = 10^{22}\ \text{m}^{-3}$, for various values of the applied field. It can be seen that for the two lower values of applied field ($E^i = 5\times10^4$ V/m and 1×10^5 V/m) the response is essentially symmetric and the $p = 1$ order current is sufficient (in these two cases only the first and first-plus-second order current is shown since higher order currents are negligible). For the

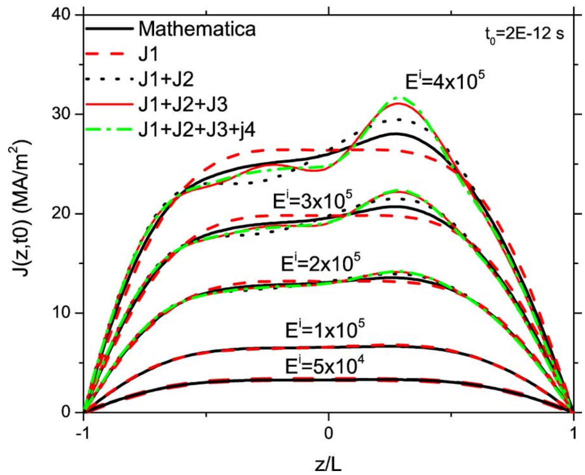

Fig. 4. Current density versus position at $t = 2$ ps and $\varepsilon_h = 1$ for various values of the applied field E^i.

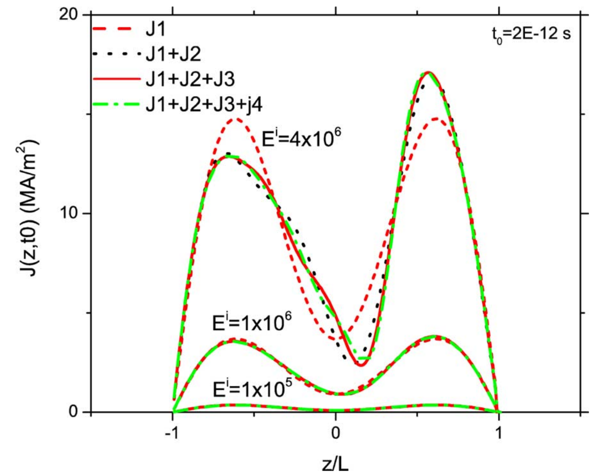

Fig. 5. Current density versus position at $t = 2$ ps and $\varepsilon_h = 12$ for various values of the applied field E^i.

three higher values of applied field, $E^i = 2, 3, 4 \times 10^5$ V/m, up to 4th order currents are shown. It is clear that for a given strength of the applied field the solution requires higher order currents, but these converge. Further, for higher field strengths the current distribution becomes asymmetric. Also shown in the plot is the solution obtained from Mathematica's NDSolve routine [23], where the three coupled equations (transport, continuity, and wave equations),

$$\left(\frac{\partial}{\partial t}+\frac{1}{\tau}\right)J_z^{\rm cond}(z,t)+\frac{\partial}{\partial z}\left(\frac{k_BT}{m}\rho(z,t)+\frac{\left(J_z^{\rm cond}(z,t)\right)^2}{\rho(z,t)}\right)$$
$$+\frac{e}{m}\rho(z,t)\left(E_z^i(z,t)+E_z^{\rm resp}(z,t)\right)=0$$
$$\frac{\partial}{\partial t}E_z^{\rm resp}(z,t)=-\frac{1}{\varepsilon}J_z^{\rm cond}(z,t)$$
$$\frac{\partial}{\partial t}\rho(z,t)+\frac{\partial}{\partial z}J_z^{\rm cond}(z,t)=0 \qquad (61)$$

were solved numerically. The main contribution to the nonlinearity is from the squared term, although the nonlinear product $\rho(z,t)E(z,t)$ contributes somewhat. In general, the trend in asymmetry of the current as field strength increases is produced in both solutions, and for moderate strength fields the agreement between the presented method and Mathematica is good. For $E^i = 4 \times 10^5$ V/m and $z < 0$ the response from the first three orders of current, $\mathbf{J}_{1-3}$, oscillates more than the Mathematica solution, but using $\mathbf{J}_{1-4}$ the solution becomes smoother. For $z > 0$ the presented method predicts a much larger peak near $z = 0.3L$. For higher values of applied field (not shown) the peak increases in both the analytical and Mathematica solution, although the analytical solution remains above the numerical Mathematica solution. The reason for the discrepancy is unclear; the Mathematica solution was stable against small changes in the numerical procedure, but nevertheless gave warnings about spatial errors being larger than tolerance (finer spatial discretization couldn't be preformed due to memory limitations). The Mathematica solver wasn't stable for $\varepsilon_h \neq 1$, which is why $\varepsilon_h = 1$ was chosen for this example.

Fig. 5 shows the same result as Fig. 4 except for $\varepsilon_h = 12$. It can be seen that the nonlinearity appears at a larger value of applied field than for $\varepsilon_h = 1$, consistent with Fig. 3, and that good convergence of the solution is again obtained.

IV. Conclusions

In this work, the nonlinear response of a semiconductor or plasma to large-amplitude, time-harmonic electromagnetic fields was obtained by solving the nonlinear transport equation using an harmonic expansion. The polarization response was assumed linear, and the conduction response was allowed to be nonlinear, and spatially and temporally dispersive. The method leads to a hierarchical set of linear second-order differential equations with non-linear forcing terms. A simple slab example was shown that admits an analytical solution, and comparison with Mathematica was shown.

Appendix I
Volume Equivalence Principle in the Time Domain for General Currents

In this appendix we show that the well-known volume-equivalence principle [13] can be applied in the time domain, including the general (non-linear, spatially- and temporally-dispersive) response of charge carriers. In [24] the authors use a similar time-domain volume-equivalence principle, although for linear, temporally-dispersive permittivities. In the following, the term dispersive refers to both spatial and temporal dispersion.

We assume a material region Ω containing nonlinear dispersive material immersed in a homogeneous simple non-dispersive medium characterized by μ_0, ε_1. We assume a current $\mathbf{J}^i$ in the exterior medium results in incident fields that interact with the medium in Ω. At any point in space we can write Maxwell's equations as

$$\nabla\times\mathbf{E}(\mathbf{r},t)=-\frac{\partial}{\partial t}\mu_0\mathbf{H}(\mathbf{r},t)$$
$$\nabla\times\mathbf{H}(\mathbf{r},t)=\frac{\partial}{\partial t}\varepsilon_0\mathbf{E}(\mathbf{r},t)+\frac{\partial}{\partial t}\mathbf{P}(\mathbf{r},t)$$
$$+\mathbf{J}^i(\mathbf{r},t)+\mathbf{J}^{\rm cond}(\mathbf{r},t) \qquad (62)$$

where we used $\mathbf{D}(\mathbf{r},t) = \varepsilon_0\mathbf{E}(\mathbf{r},t) + \mathbf{P}(\mathbf{r},t)$. For $\mathbf{r} \notin \Omega$ we have $\mathbf{P} = \varepsilon_0(\varepsilon_1/\varepsilon_0 - 1)\mathbf{E}$ and $\mathbf{J}^{\rm cond} = \mathbf{0}$, and for $\mathbf{r} \in \Omega$ we have $\mathbf{J}^i = \mathbf{0}$, and $\mathbf{P}$ and $\mathbf{J}^{\rm cond}$ are unspecified but contain the nonlinear dispersive response. Furthermore, the incident fields satisfy

$$\begin{aligned}\nabla\times\mathbf{E}^{i}(\mathbf{r},t)&=-\frac{\partial}{\partial t}\mu_0\mathbf{H}^{i}(\mathbf{r},t)\\ \nabla\times\mathbf{H}^{i}(\mathbf{r},t)&=\frac{\partial}{\partial t}\varepsilon_1\mathbf{E}^{i}(\mathbf{r},t)+\mathbf{J}^{i}(\mathbf{r},t).\end{aligned}\tag{63}$$

By adding and subtracting the term $(\partial/\partial t)\varepsilon_1\mathbf{E}(\mathbf{r},t)$ on the right side of Ampère's law in (62), we obtain

$$\begin{aligned}\nabla\times\mathbf{E}(\mathbf{r},t)=&-\frac{\partial}{\partial t}\mu_0\mathbf{H}(\mathbf{r},t)\,\nabla\times\mathbf{H}(\mathbf{r},t)\\ =&\frac{\partial}{\partial t}[\varepsilon_0-\varepsilon_1]\mathbf{E}(\mathbf{r},t)+\frac{\partial}{\partial t}\varepsilon_1\mathbf{E}(\mathbf{r},t)\\ &+\frac{\partial}{\partial t}\mathbf{P}(\mathbf{r},t)+\mathbf{J}^{i}(\mathbf{r},t)+\mathbf{J}^{\mathrm{cond}}(\mathbf{r},t)\end{aligned}\tag{64}$$

for all $\mathbf{r}$. We define the scattered fields as the difference between the total fields $\mathbf{E}$, $\mathbf{H}$ and the incident fields, $\mathbf{E}^i$, $\mathbf{H}^i$,

$$\begin{aligned}\mathbf{E}^{s}(\mathbf{r},t)&\equiv\mathbf{E}(\mathbf{r},t)-\mathbf{E}^{i}(\mathbf{r},t)\\ \mathbf{H}^{s}(\mathbf{r},t)&\equiv\mathbf{H}(\mathbf{r},t)-\mathbf{H}^{i}(\mathbf{r},t)\end{aligned}\tag{65}$$

valid for all $\mathbf{r}$; by (63) these are the fields caused by $\mathbf{J}^i$ in a homogeneous space characterized by μ_0,ε_1, i.e., in the absence of the material region Ω. Subtracting the curl (63) from (64) leads to

$$\begin{aligned}\nabla\times\mathbf{E}^{s}(\mathbf{r},t)=&-\frac{\partial}{\partial t}\mu_0\mathbf{H}^{s}(\mathbf{r},t)\\ \nabla\times\mathbf{H}^{s}(\mathbf{r},t)=&\frac{\partial}{\partial t}(\varepsilon_0-\varepsilon_1)\mathbf{E}(\mathbf{r},t)+\frac{\partial}{\partial t}\varepsilon_1\mathbf{E}^{s}(\mathbf{r},t)\\ &+\frac{\partial}{\partial t}\mathbf{P}(\mathbf{r},t)+\mathbf{J}^{\mathrm{cond}}(\mathbf{r},t).\end{aligned}\tag{66}$$

If we define equivalent current as

$$\mathbf{J}^{\mathrm{eq}}\equiv\frac{\partial}{\partial t}(\varepsilon_0-\varepsilon_1)\mathbf{E}(\mathbf{r},t)+\frac{\partial}{\partial t}\mathbf{P}(\mathbf{r},t)+\mathbf{J}^{\mathrm{cond}}(\mathbf{r},t)\tag{67}$$

which is (1), then we have

$$\begin{aligned}\nabla\times\mathbf{E}^{s}(\mathbf{r},t)&=-\frac{\partial}{\partial t}\mu_0\mathbf{H}^{s}(\mathbf{r},t)\\ \nabla\times\mathbf{H}^{s}(\mathbf{r},t)&=\frac{\partial}{\partial t}\varepsilon_1\mathbf{E}^{s}(\mathbf{r},t)+\mathbf{J}^{\mathrm{eq}}(\mathbf{r},t)\end{aligned}\tag{68}$$

valid for all $\mathbf{r}$. Thus, the scattered fields can be seen to be caused by the equivalent currents within the volume Ω, where the entire space is now homogeneous. Note that for $\mathbf{r}\notin\Omega$,

$$\begin{aligned}\mathbf{J}^{\mathrm{eq}}(\mathbf{r},t)=&\frac{\partial}{\partial t}(\varepsilon_0-\varepsilon_1)\mathbf{E}(\mathbf{r},t)\\ &+\frac{\partial}{\partial t}\varepsilon_0\left(\frac{\varepsilon_1}{\varepsilon_0}-1\right)\mathbf{E}(\mathbf{r},t)=\mathbf{0}.\end{aligned}\tag{69}$$

APPENDIX II
SATISFACTION OF DC TRANSPORT EQUATION

Here we show that the static second-order nonlinear solution (59) satisfies the dc transport equation to second order. Presumable, higher-order solutions would satisfy the dc transport equation to higher order, but this was not verified analytically.

From (3), the DC transport equation is

$$\frac{k_BT}{m}\frac{d\rho(z)}{dz}-\frac{e^2}{m}NE(z)+\frac{e}{m}\rho(z)E(z)=0.\tag{70}$$

The total electric field (this is simply the inversion of Poisson's equation applied to the inhomogeneous slab geometry, as shown below) is, using (58),

$$\begin{aligned}E(z)=&\frac{1}{\varepsilon}\int_{-L}^{z}\rho(z')\,dz'+\frac{\varepsilon_1}{\varepsilon}E^{i}(z)\\ =&\frac{\varepsilon_1}{\varepsilon}\frac{\cosh(k_Dz)}{\cosh(k_DL)}E^{i}\\ &-\frac{b_4}{\varepsilon}\left(\frac{1}{2}\sinh(2k_Dz)-\cosh(k_DL)\sinh(k_Dz)\right).\end{aligned}\tag{71}$$

Plugging into the dc transport equation and multiplying out terms, we find that first- (proportional to E^i) and second-order (proportional to $\left(E^i\right)^2$) terms identically vanish, and we are left with third and fourth order terms, $0\left(E^i\right)^3+0\left(E^i\right)^4=0$, completing the proof.

To show that the self-consistent electric field is (71), note that the static potential satisfies

$$\frac{d^2}{dz^z}\phi(z)=-\frac{\rho(z)}{\varepsilon}\tag{72}$$

where $\rho\neq0$ inside the slab. The Green's function satisfies

$$\frac{d^2}{dz^z}g(z,z')=-\delta(z-z')\tag{73}$$

leading to

$$g(z,z')=-\frac{1}{2}|z-z'|+g^h(z,z').\tag{74}$$

Therefore, the potential is

$$\phi(z)=\frac{1}{\varepsilon}\int_{-L}^{L}-\frac{1}{2}|z-z'|\rho(z')\,dz'+\phi^h(z)\tag{75}$$

where ϕ^h is a homogeneous solution of (72). To determine the scattered potential $\phi^h(z)$ and the total electric field, note that

$$\frac{d^2}{dz^z}\phi^h=0\tag{76}$$

such that

$$\phi_1^h=Az+B,\quad\phi_2^h=Fz+G,\quad\phi_3^h=Cz+D\tag{77}$$

in each region (left, inside, and right of the slab),

$$\begin{aligned}\phi_1&=\phi_1^h=Az+B\\ \phi_2&=\phi_2^p+\phi_2^h=\frac{1}{\varepsilon}\int_{-L}^{L}-\frac{1}{2}|z-z'|\rho(z')\,dz'+Fz+G\\ \phi_3&=\phi_3^h=Cz+D.\end{aligned}\tag{78}$$

Boundary conditions are

$$\phi_1=\phi_2,\quad\varepsilon_1\frac{d\phi_1}{dz}=\varepsilon\frac{d\phi_2}{dz}\tag{79}$$

at $z=L$ and

$$\phi_2=\phi_3,\quad\varepsilon\frac{d\phi_2}{dz}=\varepsilon_1\frac{d\phi_3}{dz}\tag{80}$$

at $z=-L$, and

$$\phi_{1,2}\left(|z|\rightarrow\infty\right)=-E_0 z \tag{81}$$

such that $A=C=-E_0$. This leads to $F=-\left(\varepsilon_1/\varepsilon\right)E_0$ and

$$\phi^h\left(z\right)=-\frac{\varepsilon_1}{\varepsilon}E_0 z+G \tag{82}$$

and therefore

$$\begin{aligned}E_{1,3}&=E_0\\ E_2&=-\frac{d}{dz}\phi_2\\ &=\frac{1}{\varepsilon}\frac{\partial}{\partial z}\int_{-L}^{L}\frac{1}{2}\left|z-z'\right|\rho\left(z'\right)dz'+\frac{\varepsilon_1}{\varepsilon}E_0\\ &=\frac{1}{\varepsilon}\int_{-L}^{z}\rho\left(z'\right)dz'+\frac{\varepsilon_1}{\varepsilon}E_0\end{aligned} \tag{83}$$

which is (71).

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